STOCHASTIC FLUCTUATION IN A URANIUM-ENRICHING CASCADE USING THE CENTRIFUGE PROCESS

ISOTOPES SEPARATION

KEYWORDS: *isotope separation, gas centrifuges, performance, uranium-235, gas flow, uranium hexafluoride, numerical solution, variations, enrichment*

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The parameters of a uranium-enriching cascade, i.e., the cut and the separation factor, are considered to be fluctuating stochastically. The covariance matrices of the total uranium flow and $^{235}UF_6$ *flow were derived by the classical stochastic theory for evaluating the effect of stochastic fluctuations of these parameters to steady-state plant performance. Also the stationary random process theory is applied to the kinetic equations of the cascade, and the autocorrelation function of the* $^{235}UF_6$ *flow and enrichment is derived for evaluating the time behavior of the plant performance caused by random fluctuation of these system parameters. Numerical values illustrate the response of product flow and enrichment to the fluctuations, which are both time independent and dependent, of the cut and the separation gain of stages and centrifuges. These data lead to a conclusion concerning the tolerances of centrifuge parameters and stage controllers.*

I. INTRODUCTION

A uranium-enriching plant using the centrifugal method, producing 2000 tons of 3% enriched uranium per year, is a huge cascade composed of nearly 800 000 centrifuges. A cascade of this type is a tapered cascade in which the interstage flow changes stage by stage, while the gaseous diffusion plant is often a square cascade.

The characteristics of a cascade have been discussed by Cohen[1] and many other authors[2-6]; these works have been based on deterministic analyses.

It is believed, however, that the lifetime of each centrifuge is several years, and the tolerance of its cut (i.e., the ratio of the enriched-uranium flow rate to the feed rate) is greater than the permissible tolerance in the effective cut of a separation stage. Therefore, when plant performance is to be appraised, the following two points must be considered:

Problem 1: the influence of stochastic distributions of plant parameters (that is, the cut and the separation factor) on plant performance

Problem 2: the time behavior of flow and enrichment caused by time-dependent random fluctuations of the plant parameters.

To discuss the above two problems, it does not prove sufficient to study the cascade characteristics by employing only the deterministic theory. The objectives of this paper are to evaluate these problems by the stochastic theory and to make some contribution in determining the tolerance of the centrifuge and the performance of the cascade control system. The classical stochastic theory[7] was employed to solve Problem 1 as was the cascade theory developed by Cohen, and the theory of the stationary random process[8] was used to solve Problem 2. It was also necessary to extend the method of treating the dynamics of a tapered cascade derived by Higashi, Oya, and Oishi[6] for Problem 2.

II. THE EQUATIONS OF THE CASCADE

Statics

Considered is an isotope separation cascade which consists of $N + 1$ stages in the rectifier side and M stages in the stripper side. The interstage flow rates at the i'th stage are illustrated in Fig. 1, where F_i, G_i, and N_i are the total uranium flow rate ($^{235}UF_6$ + $^{238}UF_6$), the ^{235}U flow rate ($^{235}UF_6$), and the enrichment (G_i/F_i) of the i'th-stage feed. (The primes $'$ and $''$ indicate the enriched and depleted streams, respectively.)

The statics equations at the i'th stage are

$$F_i' = \theta_i F_i \tag{1}$$

$$\frac{N_i'}{1 - N_i'} = \gamma_i \frac{N_i''}{1 - N_i''} \tag{2}$$

$$F_i' + F_i'' = F_i \tag{3}$$

$$F_i'N_i' + F_i''N_i'' = F_i N_i \text{ or } G_i' + G_i'' = G_i \quad , \tag{4}$$

where θ_i and γ_i are the cut and the separation factor of the i'th separation stage, respectively. Postulating the condition

$$\gamma_i \approx 1 \quad , \quad N_i' << 1 \quad , \quad N_i'' << 1 \quad , \tag{5}$$

the approximate equations of the enrichment can be derived from Eqs. (1) through (4); that is,

$$N_i' = N_i \{1 + 2(1 - \theta_i)\epsilon_i\} \tag{6}$$

$$N_i'' = N_i \{1 - 2\theta_i \epsilon_i\} \quad , \tag{7}$$

where ϵ_i is the separation gain defined by

$$\gamma_i \equiv 1 + 2\epsilon_i \quad . \tag{8}$$

The flow rate equations of the cascade at the steady state are obtained from the material balance around stage i (see Fig. 1):

$$F_i = \theta_{i-1}F_{i-1} + (1 - \theta_{i+1})F_{i+1} + \delta_{i,0}F_f \quad , \tag{9}$$

$$\begin{aligned} G_i = {} & \theta_{i-1}\{1 + 2(1 - \theta_{i-1})\epsilon_{i-1}\}G_{i-1} \\ & + (1 - \theta_{i+1})\{1 - 2\theta_{i+1}\epsilon_{i+1}\}G_{i+1} \\ & + \delta_{i,0}F_f N_f \end{aligned} \tag{10}$$

$$i = -M, -M + 1, \ldots, 0, 1, \ldots, N \quad .$$

In these equations, F_f and N_f are the feed uranium flow rate and its enrichment supplied to the feed stage ($i = 0$), and $\delta_{i,0}$ is the Kronecker's δ function. As shown in Fig. 2, the downstreams into the top stage ($i = N$), i.e., the second terms of Eqs. (9) and (10), are equal to zero; likewise, the upstreams, i.e., the first terms, are equal to zero at the bottom stage ($i = -M$). The product flow rates F_P and G_P are the N'th-stage enriched flow rates:

$$F_P = \theta_N F_N \quad , \tag{11}$$

$$G_P = \theta_N\{1 + 2(1 - \theta_N)\epsilon_N\}G_N \quad , \tag{12}$$

and the waste flow rates F_W and G_W are the $-M$'th-stage depleted flow rates:

$$F_W = (1 - \theta_{-M})F_{-M} \quad , \tag{13}$$

$$G_W = (1 - \theta_{-M})(1 - 2\theta_{-M}\epsilon_{-M})G_{-M} \quad . \tag{14}$$

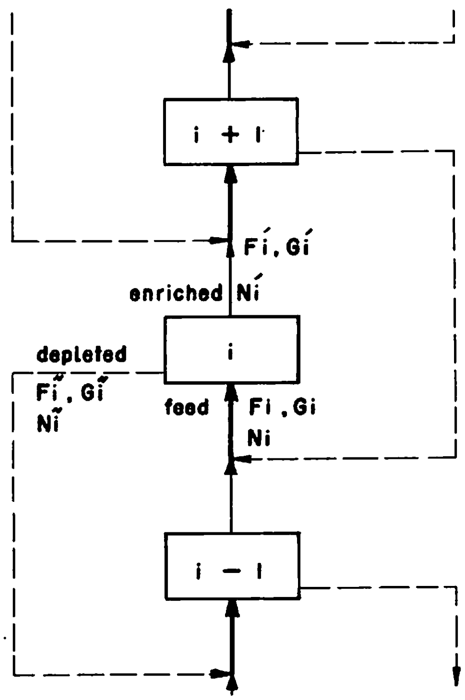

Fig. 1. Interstage flow rates at the i'th stage of the cascade.

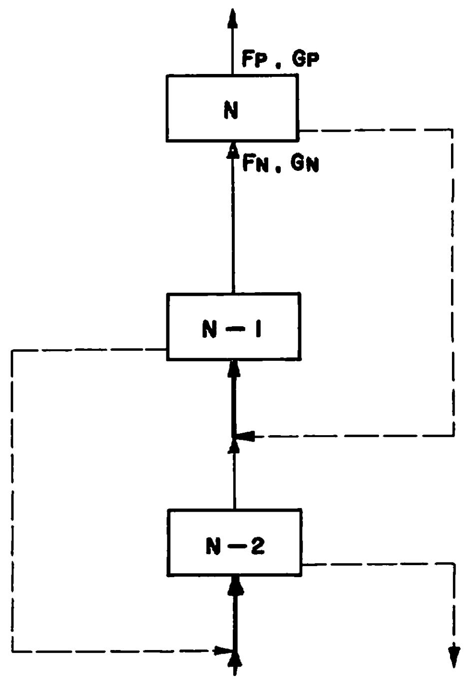

Fig. 2. Flow rates at the top stage.

It is desirable that the cut θ_i be determined to satisfy the condition of no-mixing loss, i.e., the condition of the ideal cascade

$$\theta_i = \frac{1 + (\sqrt{\gamma_i} - 1)N_i}{1 + \sqrt{\gamma_i}} \quad . \tag{15}$$

Generally the separation factor is the same for every centrifuge, and, under the condition of Eq. (5), Eq. (15) is expressed approximately as

$$\theta = \frac{1}{2} - \frac{\epsilon}{4} \quad , \tag{16}$$

which is constant and independent of the stage. The cascade with a constant cut, termed a tapered cascade, is nearly ideal when the cut is set as in Eq. (16). The cascade treated here is of this type, and relations which hold for the ideal cascade are used when necessary.

Dynamics

The kinetic equations for the total uranium flow rate F_i and the ^{235}U flow rate $F_i N_i$ or G_i are given by

$$T\frac{dF_i(t)}{dt} + F_i(t) - \theta_{i-1}(t)F_{i-1}(t) - \{1 - \theta_{i+1}(t)\}F_{i+1}(t)$$
$$= \delta_{i,0}\left\{T\frac{dF_f(t)}{dt} + F_f(t)\right\} \quad , \tag{17}$$

$$T\frac{dG_i(t)}{dt} + G_i(t)$$
$$- \theta_{i-1}(t)\left[1 + 2\{1 - \theta_{i-1}(t)\}\epsilon_{i-1}(t)\right]G_{i-1}(t)$$
$$- \{1 - \theta_{i+1}(t)\}\{1 - 2\theta_{i+1}(t)\epsilon_{i+1}(t)\}G_{i+1}(t)$$
$$= \delta_{i,0}\left\{T\frac{dF_f(t)N_f(t)}{dt} + F_f(t)N_f(t)\right\} \quad , \tag{18}$$

where T is the holdup time of a stage.

It is necessary to transform the above equations which are discrete in stage into space-continuous types. This is required in the analysis of the stationary stochastic treatment in Sec. IV.

Rearranging the terms in Eq. (17), the following difference equation is obtained for total uranium flow rates:

$$T\frac{dF_i(t)}{dt} - \frac{1}{2}\{F_{i-1}(t) - 2F_i(t) + F_{i+1}(t)\}$$
$$- \frac{1}{2}\{F_{i+1}(t) - F_{i-1}(t)\}$$
$$+ \{\theta_{i+1}(t)F_{i+1}(t) - \theta_{i-1}(t)F_{i-1}(t)\}$$
$$= \delta_{i,0}\left\{T\frac{dF_f(t)}{dt} + F_f(t)\right\} \quad . \tag{19}$$

Equation (19) is converted to the partial differential equation by taking the continuous limit in stage i:

$$T\frac{\partial F(\chi,t)}{\partial t} - \frac{1}{2}\frac{\partial^2 F(\chi,t)}{\partial \chi^2} - \frac{\partial F(\chi,t)}{\partial \chi} + 2\frac{\partial \theta(\chi,t)F(\chi,t)}{\partial \chi}$$
$$= \delta(\chi - \chi_0)\left\{T\frac{\partial F_f(t)}{\partial t} + F_f(t)\right\} \quad , \tag{20}$$

where χ is the coordinate which represents the stage number, as shown in Fig. 3. The range of χ is $[0, l]$, where $l = M + N + 2$, and χ_0 is the feed stage; i.e., $\chi_0 = M + 1$. The boundary condition of Eq. (20) is

$$F(0,t) = F(l,t) = 0 \quad . \tag{21}$$

Likewise, Eq. (18) for ^{235}U flow rates is transformed into

$$T\frac{\partial G(\chi,t)}{\partial t} - \frac{1}{2}\frac{\partial^2 G(\chi,t)}{\partial \chi^2} - \frac{\partial G(\chi,t)}{\partial \chi}$$
$$+ 2\frac{\partial G(\chi,t)\theta(\chi,t)}{\partial \chi}$$
$$+ 4\frac{\partial G(\chi,t)\epsilon(\chi,t)\theta(\chi,t)\{1 - \theta(\chi,t)\}}{\partial \chi}$$
$$= \delta(\chi - \chi_0)\left\{T\frac{\partial F_f(t)N_f(t)}{\partial t} + F_f(t)N_f(t)\right\} \quad , \tag{22}$$

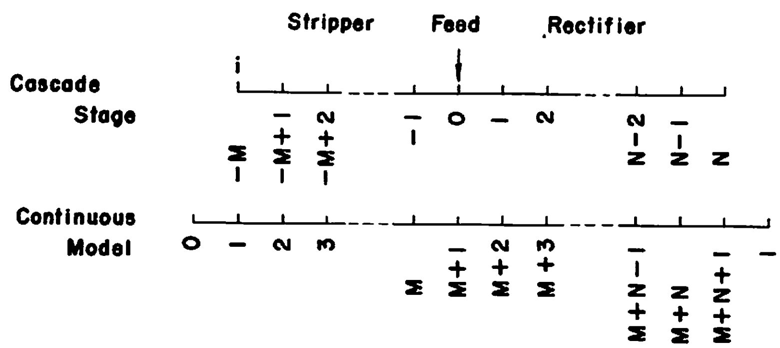

Fig. 3. Correspondence of stage numbers i and χ.

with the boundary condition

$$G(0,t) = G(l,t) = 0 \quad . \tag{23}$$

In Sec. IV we must derive the impulse response functions of the cascade, where the input is the deviation of the i'th-stage parameters (θ,ϵ), and the output is the k'th-stage flow rate deviation or enrichment deviation. Therefore, it is convenient to linearize Eqs. (20) and (22) in order to treat the fluctuations of the system parameters (θ,ϵ) as the source terms.

The variational equation for the total uranium flow rate is derived from Eq. (20):

$$T\,\frac{\partial\delta F(X,t)}{\partial t} - \frac{1}{2}\,\frac{\partial^2\delta F(X,t)}{\partial X^2} - \frac{\partial\delta F(X,t)}{\partial X} + 2\,\frac{\partial\theta(X)\delta F(X,t)}{\partial X}$$
$$= -2\,\frac{\partial F(X)\delta\theta(X,t)}{\partial X} + \delta(X - X_0)\left\{T\,\frac{\partial\delta F_f(t)}{\partial t} + \delta F_f(t)\right\} \quad . \tag{24}$$

Using Eq. (16), Eq. (24) is further reduced to

$$T\,\frac{\partial\delta F(X,t)}{\partial t} - \frac{1}{2}\,\frac{\partial^2\delta F(X,t)}{\partial X^2} - \frac{\epsilon}{2}\,\frac{\partial\delta F(X,t)}{\partial X}$$
$$= -2\,\frac{\partial F(X)\delta\theta(X,t)}{\partial X} + \delta(X - X_0)\left\{T\,\frac{\partial\delta F_f(t)}{\partial t} + \delta F_f(t)\right\} \quad . \tag{25}$$

In the case of enrichment, the variational equation of Eq. (22) can be simplified by using Eqs. (16) and (25), and the relations

$$\frac{\partial N}{\partial X} = \epsilon N \quad , \quad \frac{\partial^2 N}{\partial X^2} = \epsilon^2 N \quad , \tag{26}$$

which are the continuous limits of the following enrichment relations in the ideal cascade:

$$N_{i+1} - N_{i-1} = 2\epsilon_i N_i$$
$$N_{i+1} - 2N_i + N_{i-1} = \epsilon^2 N_i \quad . \tag{27}$$

The result is

$$T\,\frac{\partial F(X)\delta N(X,t)}{\partial t} - \frac{1}{2}\,\frac{\partial^2 F(X)\delta N(X,t)}{\partial X^2} + \frac{\epsilon}{2}\,\frac{\partial F(X)\delta N(X,t)}{\partial X}$$
$$= -\,\frac{\partial F(X)N(X)\delta\epsilon(X,t)}{\partial X} - 2F(X)\,N(X)\,\epsilon\,\delta\theta(X,t)$$
$$+ \delta(X - X_0)\left\{T\,\frac{\partial F_f\,\delta N_f(t)}{\partial t} + F_f\,\delta N_f(t)\right\} \quad . \tag{28}$$

The validity of the continuous model will be assured by directly comparing the results with those of the original discrete cascade equations.

III. THE METHOD OF EVALUATING STATIC DEVIATIONS

Total Flow

By employing the matrix-vector notation, uranium total flow balance Eqs. (9), (11), and (13) are expressed as

$$\Theta F + S = 0 \quad , \tag{29}$$

where the matrix Θ and vectors F and S are defined as

$$\Theta \equiv \begin{bmatrix} -1 & & \theta_N & & & & & \\ & -1 & & \theta_{N-1} & & & & \\ 1-\theta_N & & -1 & & \theta_{N-Z} & & 0 & \\ & \ddots & & \ddots & & \ddots & & \\ & & 1-\theta_1 & & -1 & & \theta_{-1} & \\ & & & \ddots & & \ddots & & \ddots \\ 0 & & & 1-\theta_{-M+Z} & & -1 & & \theta_{-M} \\ & & & & 1-\theta_{-M+1} & & -1 & \\ & & & & & 1-\theta_M & & -1 \end{bmatrix} \tag{30}$$

$$F \equiv [F_P\,, F_N\,, F_{N-1}, \ldots, F_0, \ldots, F_{-M+1}, F_{-M}\,, F_w\,]^T \quad , \tag{31}$$

$$S \equiv [0,0,0,\ldots,F_f\,,\ldots,0,0,0]^T \quad , \tag{32}$$

where $[\ \ ]^T$ is the transpose of $[\ \ ]$.

Assume a situation where the cut θ_i deviates at random because of a manufacturing error or for some other reason. In this case, the matrix Θ is expressed as the sum of the nominal matrix Θ_0 and the deviation matrix θ; that is,

$$\Theta = \Theta_0 + \theta \quad , \tag{33}$$

where the expectation of the matrix θ, $E\langle\theta\rangle$, is assumed to be the zero matrix. Then the flow vector F is also written as the sum of the expectation F_0 and the deviation f, as

$$F = F_0 + f \quad . \tag{34}$$

These two terms can be obtained by substituting Eqs. (33) and (34) into Eq. (29) and neglecting the second-order term θf, as

$$F_0 \equiv E\langle F\rangle = -\Theta_0^{-1}S \tag{35}$$

$$f = -\Theta_0^{-1}\theta F_0 \quad . \tag{36}$$

The expectation of the flow deviation f is equal to the zero vector because of the assumption $E\langle\theta\rangle$. The covariance matrix ϕ_{ff}, defined as the expectation of the product of f and f^T, is

$$\phi_{ff} = \Theta_0^{-1}E\langle\theta F_0 F_0^T\theta^T\rangle\,\Theta_0^{-1T} \quad . \tag{37}$$

Postulating that there are no interstage correlations of the cut deviations, i.e.,

$$E\langle \delta\theta_i \ \delta\theta_j \rangle = \delta_{i,j} \ \sigma_{\theta_i}^2 \quad , \tag{38}$$

where σ_{θ_i} is the standard deviation of cut at the stage i, the matrix $E\langle \theta F_0 F_0^T \theta^T \rangle$ in Eq. (37) is derived as

$$\begin{bmatrix} F_N^2\sigma_N^2 & 0 & -F_N^2\sigma_N^2 & & & & & & \\ 0 & F_{N-1}^2\sigma_{N-1}^2 & 0 & -F_{N-1}^2\sigma_{N-1}^2 & & & & 0 & \\ -F_N^2\sigma_N^2 & 0 & F_{N-2}^2\sigma_{N-2}^2 + F_N^2\sigma_N^2 & 0 & -F_{N-2}^2\sigma_{N-2}^2 & & & & \\ & -F_{N-1}^2\sigma_{N-1}^2 & 0 & F_{N-3}^2\sigma_{N-3}^2 + F_{N-1}^2\sigma_{N-1}^2 & 0 & -F_{N-3}^2\sigma_{N-3}^2 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & 0 & & -F_{-M+2}^2\sigma_{-M+2}^2 & 0 & F_{-M}^2\sigma_{-M}^2 + F_{-M+2}^2\sigma_{-M+2}^2 & 0 & -F_{-M}^2\sigma_{-M}^2 & \\ & & & & -F_{-M+1}^2\sigma_{-M+1}^2 & 0 & F_{-M+1}^2\sigma_{-M+1}^2 & 0 & \\ & & & & & -F_{-M}^2\sigma_{-M}^2 & 0 & F_{-M}^2\sigma_{-M}^2 & \end{bmatrix} . \tag{39}$$

The square root of the diagonal elements of the covariance matrix ϕ_{ff} are the standard deviations of the uranium total flow deviations caused by the random distribution of cut.

$^{235}UF_6$ Flow

In the case of $^{235}UF_6$ flow rate, some approximations are made to simplify the following discussions. Because the cut θ_i is nearly equal to 0.5, an approximate equation can be derived:

$$[2\theta(1-\theta)]_{\theta=\theta_0+\delta\theta} = 0.5\left(1 - \frac{\epsilon^2}{4}\right) + \epsilon\delta\theta$$
$$\approx 0.5 \quad . \tag{40}$$

With this approximation, the $^{235}UF_6$ balance Eqs. (10), (12), and (14) can be reduced to

$$G_i = (\theta_{i-1} + 0.5\,\epsilon_{i-1})G_{i-1} + \{1 - (\theta_{i+1} + 0.5\,\epsilon_{i+1})\}G_{i+1} + \delta_{i,0}F_f N_f \quad , \tag{41}$$

$$G_P = (\theta_N + 0.5\,\epsilon_N)G_N \quad , \tag{42}$$

$$G_W = \{1 - (\theta_{-M} + 0.5\,\epsilon_{-M})\}G_{-M} \quad . \tag{43}$$

Expressing the balance equations by the matrix-vector notation

$$\Gamma G + U = 0 \tag{44}$$

and writing Γ and G as

$$\Gamma = \Gamma_0 + \gamma \quad , \tag{45}$$

$$G = G_0 + g \quad , \tag{46}$$

the expectation of G, $E\langle G \rangle \equiv G_0$, the deviation and the covariance matrix ϕ_{gg} are derived in a similar manner as formerly:

$$G_0 = -\Gamma_0^{-1} U \tag{47}$$

$$g = -\Gamma_0^{-1}\gamma G_0 \tag{48}$$

$$\phi_{gg} = \Gamma_0^{-1} E\langle \gamma G_0 G_0^T \gamma^T \rangle \, \Gamma_0^{-1^T} \quad . \tag{49}$$

Assuming no correlation of the fluctuations of the separation gain $\delta\epsilon_i$ and the cut $\delta\theta_j$,

$$E\langle \delta\epsilon_i \, \delta\theta_j \rangle = 0 \quad , \tag{50}$$

$$E\langle \delta\theta_i \, \delta\theta_j \rangle = \delta_{i,j} \, \sigma_{\theta_i}^2 \quad , \tag{51}$$

$$E\langle \delta\epsilon_i \, \delta\epsilon_j \rangle = \delta_{i,j} \, \sigma_{\epsilon_i}^2 \quad , \tag{52}$$

the matrix $E\langle \gamma G_0 G_0^T \gamma^T \rangle$ in Eq. (49) is given by replacing $F_i^2\sigma_{\theta_i}^2$ with $G_i^2(\sigma_{\theta_i}^2 + 0.25\,\sigma_{\epsilon_i}^2)$ in Eq. (39).

Enrichment

The enrichment N_i is defined as

$$N_i = \frac{G_i}{F_i} \quad . \tag{53}$$

The expectation of the squared enrichment deviation n_i, $E\langle n_i^2 \rangle$, caused by the random deviations of the cut and the separation factor, is

$$E\langle n_i^2 \rangle = \left(\frac{G_i}{F_i^2}\right)^2 E\langle ff_{ii} \rangle + \left(\frac{1}{F_i}\right)^2 E\langle gg_{ii} \rangle - 2\,\frac{G_i}{F_i^3} E\langle f_i\, g_i \rangle \quad , \tag{54}$$

where $E\langle ff_{ii} \rangle$ and $E\langle gg_{ii} \rangle$ are the i'th diagonal elements of the matrices ϕ_{ff} and ϕ_{gg}. The expectation $E\langle f_i\, g_i \rangle$ is the i'th diagonal element of the matrix $E\langle fg^T \rangle$; that is,

$$E\langle fg^T \rangle = \Theta_0^{-1} E\langle \theta F_0 G_0^T \gamma^T \rangle \, \Gamma_0^{-1^T} \quad , \tag{55}$$

where the matrix $E\langle \theta F_0 G_0^T \gamma^T \rangle$ is obtained by replacing $F_i^2\sigma_i^2$ with $F_i\, G_i \sigma_{\theta_i}^2$ in Eq. (39), under the conditions of Eqs. (50), (51), and (52).

IV. THE METHOD OF EVALUATING DYNAMIC FLUCTUATION

Formulas

First some basic formulas of stochastic process theory are presented which are used in the following formulations.

Let $\{X_i(t)\}$ denote the stationary stochastic process in the wide sense; i.e.,

$$E\langle X_i(t)\rangle = \mu_i < \infty \quad , \tag{56}$$

$$E\langle\{X_i(t_1) - \mu_i\}\{X_j(t_2) - \mu_j\}\rangle = \phi_{ij}(|t_1 - t_2|) \quad , \tag{57}$$

where the mean value μ_i is independent of time t, and the coorelation function $\phi_{ij}(t)$ depends only on the time difference $|t_1 - t_2|$. In this paper, only the particular case of $\mu_i = 0$ is discussed.

The relation between the correlation function $\phi_{ij}(t)$ and the spectral power density function $\Phi_{ij}(j\omega)$ ($i = j$, autocorrelation; $i \neq j$, cross correlation) is written as

$$\Phi_{ij}(j\omega) = \int_{-\infty}^{\infty}\phi_{ij}(t)\exp(-j\omega t)dt \tag{58}$$

$$\phi_{ij}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{ij}(j\omega)\exp(j\omega t)d\omega \quad . \tag{59}$$

When the process $\{Y(t)\}$ consists of two processes $\{X_1(t)\}$ and $\{X_2(t)\}$,

$$Y(t) = X_1(t) + X_2(t) \quad , \tag{60}$$

the functions $\phi_{YY}(t)$ and $\Phi_{YY}(j\omega)$ are expressed as

$$\phi_{YY}(t) = \phi_{11}(t) + \phi_{12}(t) + \phi_{21}(t) + \phi_{22}(t) \tag{61}$$

$$\Phi_{YY}(j\omega) = \Phi_{11}(j\omega) + \Phi_{12}(j\omega) + \Phi_{21}(j\omega) + \Phi_{22}(j\omega) \quad . \tag{62}$$

If the processes $\{X_1(t)\}$ and $\{X_2(t)\}$ are independent of each other, $\phi_{ij}(t)$ and $\Phi_{ij}(j\omega)$ for $i \neq j$ are equal to zero.

Now consider the linear filter G, illustrated in Fig. 4. The transfer function $G(j\omega)$ is obtained by Fourier transformation of the impulse response $g(t)$ of the filter:

$$G(j\omega) = \int_{-\infty}^{\infty} g(t)\exp(-j\omega t)dt \quad . \tag{63}$$

The output process $\{Y(t)\}$ corresponding to the input process $\{X(t)\}$, which is the stationary stochastic process in the wide sense, is also the stochastic process. Its mean value μ_Y, autocorrelation function $\phi_{YY}(t)$, and spectral power density $\Phi_{YY}(j\omega)$ are obtained as

$$\mu_Y = \mu_X \int_{-\infty}^{\infty} g(t)dt \quad , \tag{64}$$

$$\phi_{YY}(t) = \int_{-\infty}^{\infty} g(t_1)\int_{-\infty}^{\infty} g(t_2)\phi_{XX}(t - t_1 + t_2)dt_2dt_1 \quad , \tag{65}$$

$$\Phi_{YY}(j\omega) = G(j\omega)\,G(-j\omega)\,\Phi_{XX}(j\omega) \quad . \tag{66}$$

Fig. 4. Linear filter G.

Procedures

The following assumptions are made in this paper:

1. The fluctuations of the cut θ_i and the separation gain ϵ_i from the nominal values $\delta\theta_i(t)$ and $\delta\epsilon_i(t)$ are the stationary stochastic processes with zero mean values.
2. There are no interstage correlations of fluctuations $\delta\theta_i(t)$ and $\delta\epsilon_i(t)$. The fluctuations of the cut and the separation gain have finite variance and are not dependent on each other.
3. The relation between the fluctuation of the flow rate and the fluctuation of the cut or the separation gain is described by the transfer function.

Under the above assumptions, the procedure for evaluating the fluctuations of the total flow rate and the enrichment caused by random fluctuations of the cascade parameters (θ_i, ϵ_i) is shown in Fig. 5. In this figure, $g_{ik}(t)$ is the impulse response of the k'th-stage flow or enrichment to the i'th-stage parameter disturbance, and $\phi_i^N(t)$ is the autocorrelation function of the i'th-stage random disturbance of the system parameters.

The Transfer Function of the Cascade

The impulse responses of the cascade can be derived from Eqs. (25) and (28). Employing a nondimensional time τ,

$$\tau \equiv \frac{t}{2T} \quad , \tag{67}$$

where T is the holdup time of a stage, these equations can be reduced to

$$\frac{\partial\delta F(\chi,\tau)}{\partial\tau} - \frac{\partial^2\delta F(\chi,\tau)}{\partial\chi^2} - \epsilon\frac{\partial\delta F(\chi,\tau)}{\partial\chi} = -4\,\frac{\partial F(\chi)\delta\theta(\chi,\tau)}{\partial\chi} \quad , \tag{68}$$

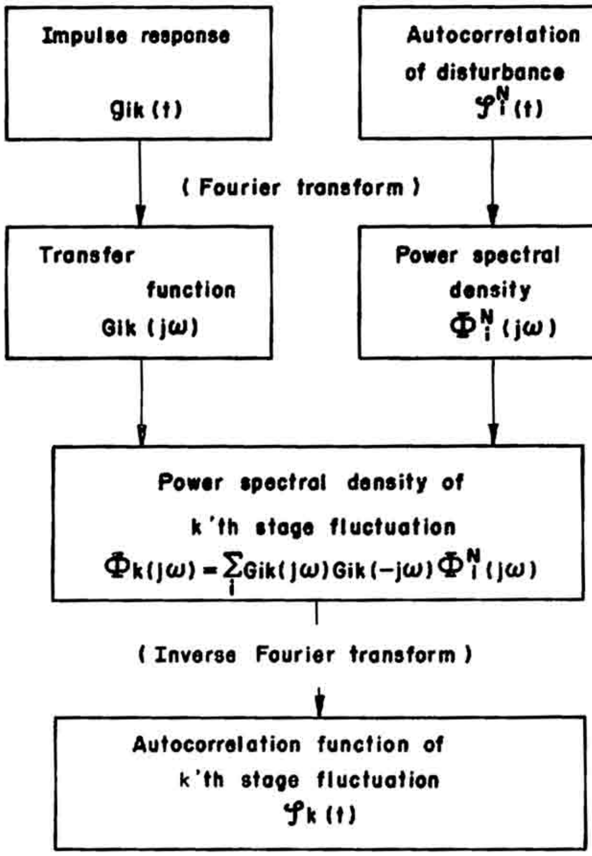

Fig. 5. Procedures for evaluating dynamic fluctuation.

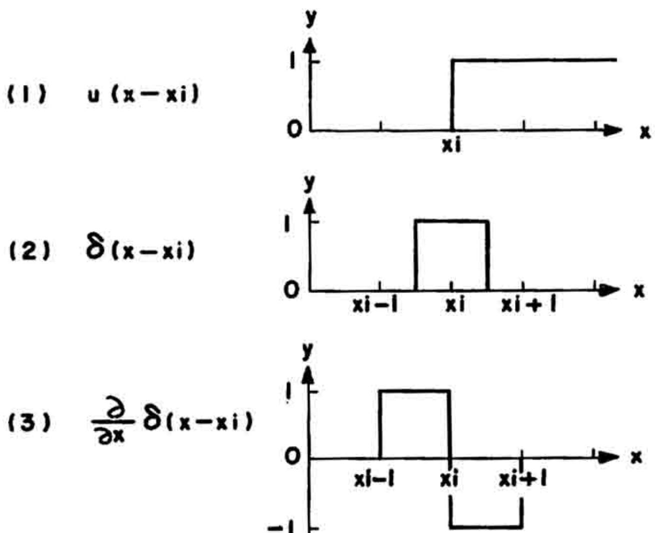

Fig. 6. Forms of deviation used to calculate impulse responses.

$$\frac{\partial \delta G(\chi,\tau)}{\partial \tau} - \frac{\partial^2 \delta G(\chi,\tau)}{\partial \chi^2} + \epsilon \frac{\partial \delta G(\chi,\tau)}{\partial \chi} = -2\,\frac{\partial G(\chi)\delta\epsilon(\chi,\tau)}{\partial \chi} - 4\epsilon G(\chi)\delta\theta(\chi,\tau) \quad , \tag{69}$$

where

$$\delta G(\chi,\tau) = F(\chi)\delta N(\chi,\tau) \quad .$$

To obtain the impulse response functions, the i'th-stage disturbances $\delta\theta(\chi_i,\tau)$ and $\frac{\partial \delta\theta(\chi_i,\tau)}{\partial \chi}$ or $\frac{\partial \delta\epsilon(\chi_i,\tau)}{\partial \chi}$ are approximated in space to assume forms (2) and (3) in Fig. 6. Then the impulse responses to these disturbances are calculated by

Form (2): $R(\chi_i - \frac{1}{2},\tau) - R(\chi_i + \frac{1}{2},\tau)$

Form (3): $R(\chi_i - 1,\tau) - 2R(\chi_i,\tau) + R(\chi_i + 1,\tau)$,

where $R(\chi_i,\tau)$ is the impulse response to the input of form (1) in Fig. 6.

Forms and notations of the response functions for various input-output relations are

	Impulse Response	Input	Output	Form
(i)	$g_{ik}^{\theta F}(\tau)$	$\delta\theta(\chi_i,\tau)$	$\delta F(\chi_k,\tau)$	(3)
(ii)	$g_{ik}^{\theta N}(\tau)$	$\delta\theta(\chi_i,\tau)$	$\delta G(\chi_k,\tau)$	(2)
(iii)	$g_{ik}^{\epsilon N}(\tau)$	$\delta\epsilon(\chi_i,\tau)$	$\delta G(\chi_k,\tau)$	(3) .

The solution of the partial differential equation

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial \chi^2} \pm \epsilon\,\frac{\partial y}{\partial \chi} + \delta(t-0)\,U(\chi - \chi_i) \tag{70}$$

is

$$y^{\pm}(\chi_i;\chi,t) = \sum_{n=1}^{\infty} \exp(-\lambda_n\tau)\,Z_n^{\pm}(\chi_i;\chi) \tag{71}$$

$$Z_n^{\pm}(\chi_i;\chi) \equiv \frac{2}{l}\left\{\mp\frac{\epsilon}{2\lambda_n}\sin\left(\frac{n\pi\chi_i}{l}\right) + \frac{n\pi}{l\lambda_n}\cos\left(\frac{n\pi\chi_i}{l}\right)\right\} \times \exp\left[\pm\frac{\epsilon}{2}(\chi_i - \chi)\right]\sin\left(\frac{n\pi\chi}{l}\right) \tag{72}$$

$$\lambda_n \equiv \left(\frac{n\pi}{l}\right)^2 + \left(\frac{\epsilon}{2}\right)^2 \quad . \tag{73}$$

Therefore, the impulse response functions are

(i)
$$g_{ik}^{\theta F}(\tau) = \sum_{n=1}^{\infty} a_{nik}\exp(-\lambda_n\tau) \tag{74}$$

$$a_{nik} = -4F(\chi_i)\,\{Z_n^{+}(\chi_i - 1;\chi_k) - 2Z_n^{+}(\chi_i;\chi_k) + Z_n^{+}(\chi_i + 1;\chi_k)\} \quad , \tag{75}$$

(ii)
$$g_{ik}^{\theta N}(\tau) = \sum_{n=1}^{\infty} b_{nik}\exp(-\lambda_n\tau) \tag{76}$$

$$b_{nik} = -4\epsilon G(\chi_i)\left\{Z_n^-(\chi_i - \tfrac{1}{2};\chi_k) - Z_n^-(\chi_i + \tfrac{1}{2};\chi_k)\right\} \quad , \tag{77}$$

(iii)
$$g_{ik}^{\epsilon N}(\tau) = \sum_{n=1}^{\infty} c_{nik}\exp(-\lambda_n\tau) \tag{78}$$

$$c_{nik} = -2G(\chi_i)\left\{Z_n^-(\chi_i - 1;\chi_k) - 2Z_n^-(\chi_i;\chi_k) + Z_n^-(\chi_i + 1;\chi_k)\right\} \quad . \tag{79}$$

The transfer functions are obtained by a Fourier transformation of Eqs. (74), (76), and (78).

(i)
$$G_{ik}^{\theta F}(j\omega) = \sum_{n=1}^{\infty}\frac{a_{nik}}{j\omega + \lambda_n} \quad , \tag{80}$$

(ii)
$$G_{ik}^{\theta N}(j\omega) = \sum_{n=1}^{\infty}\frac{b_{nik}}{j\omega + \lambda_n} \quad , \tag{81}$$

(iii)
$$G_{ik}^{\epsilon N}(j\omega) = \sum_{n=1}^{\infty}\frac{c_{nik}}{j\omega + \lambda_n} \quad . \tag{82}$$

The Spectral Power Density of the Disturbance

Let the random fluctuation of the cut or separation gain be a stationary random rectangular pulse series, as shown in Fig. 7. The stochastic distribution of the pulse height is assumed to be the normal distribution with zero mean value:

$$P(r) = \frac{1}{\sqrt{2\pi\sigma_i^2}}\exp\left(-\frac{r^2}{2\sigma_i^2}\right) \quad . \tag{83}$$

Further, assuming the frequency of the change to follow the Poisson distribution, the pulse width l distributes according to the exponential distribution with the mean width $1/\beta_i$; that is,

$$H(l) = \beta_i\exp(-\beta_i l) \quad . \tag{84}$$

The autocorrelation function of the i'th-stage disturbance $\phi_i^N(\tau)$ is

$$\phi_i^N(\tau) = \sigma_i^2\exp(-\beta_i|\tau|) \quad , \tag{85}$$

and the spectral power density function $\Phi_i^N(j\omega)$ is obtained as the Fourier transformation of $\phi_i^N(\tau)$:

$$\Phi_i^N(j\omega) = \frac{2\beta_i\sigma_i^2}{\omega^2 + \beta_i^2} \quad . \tag{86}$$

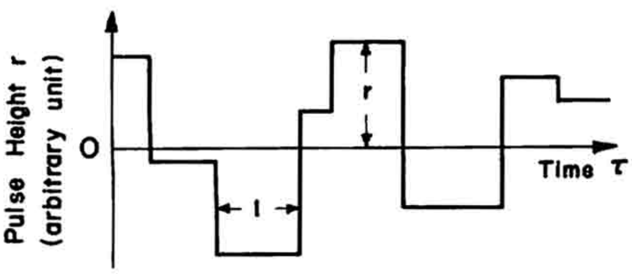

Fig. 7. Stationary pulse series of disturbance.

The Autocorrelation Function of Flow Fluctuation

The spectral power density function of the k'th-stage total flow fluctuation, caused by the random disturbance of the cut, is

$$\Phi_k^{\theta F}(j\omega) = \sum_{i=-M}^{N} G_{ik}^{\theta F}(j\omega)\,G_{ik}^{\theta F}(-j\omega)\,\Phi_i^N(j\omega)$$
$$= \sum_{i=-M}^{N}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{2\beta_{\theta_i}\sigma_{\theta_i}^2 a_{nik}a_{mik}}{(j\omega + \lambda_n)(-j\omega + \lambda_m)(\omega^2 + \beta_{\theta_i}^2)} \quad . \tag{87}$$

Then the autocorrelation function $\phi_k^{\theta F}(\tau)$ is obtained by the inverse Fourier transformation of Eq. (87).

$$\phi_k^{\theta F}(\tau) = \sum_{i=-M}^{N}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} 2\beta_{\theta_i}\sigma_{\theta_i}^2 a_{nik}a_{mik}w_{nm}(\tau) \quad , \tag{88}$$

$$w_{nm}(\tau) = \begin{cases} \dfrac{1}{2(\lambda_n + \lambda_m)}\left\{\dfrac{\exp(-\lambda_n|\tau|)}{\beta_{\theta_i}^2 - \lambda_n^2} + \dfrac{\exp(-\lambda_m|\tau|)}{\beta_{\theta_i}^2 - \lambda_m^2}\right\} + \dfrac{(\lambda_n\lambda_m - \beta_{\theta_i}^2)\exp(-\beta_{\theta_i}|\tau|)}{2\beta_{\theta_i}(\beta_{\theta_i}^2 - \lambda_n^2)(\beta_{\theta_i}^2 - \lambda_m^2)} \;, & \lambda_n \neq \beta_{\theta_i} \;,\; \lambda_m \neq \beta_{\theta_i} \\[2ex] \dfrac{\exp(-\lambda_m|\tau|)}{2(\beta_{\theta_i}^2 - \lambda_m^2)(\beta_{\theta_i} + \lambda_m)} + \left\{\dfrac{1 + (\beta_{\theta_i} + \lambda_m)|\tau|}{4\beta_{\theta_i}(\beta_{\theta_i} + \lambda_m)^2} - \dfrac{\lambda_m}{4\beta_{\theta_i}^2(\beta_{\theta_i}^2 - \lambda_m^2)}\right\}\exp(-\beta_{\theta_i}|\tau|) \;, & \lambda_n = \beta_{\theta_i} \;,\; \lambda_m \neq \beta_{\theta_i} \\[2ex] \dfrac{1 + \beta_{\theta_i}|\tau|}{4\beta_{\theta_i}^3}\exp(-\beta_{\theta_i}|\tau|) \;, & \lambda_n = \beta_{\theta_i} \;,\; \lambda_m = \beta_{\theta_i} \quad . \end{cases} \tag{89}$$

Likewise, the autocorrelation functions $\phi_k^{\theta N}(\tau)$ and $\phi_k^{\epsilon N}(\tau)$ are obtained by replacing $(\beta_{\theta_i}, \sigma_{\theta_i}, a_{nik})$ with $(\beta_{\theta_i}, \sigma_{\theta_i}, b_{nik})$ and $(\beta_{\epsilon_i}, \sigma_{\epsilon_i}, c_{nik})$, respectively.

V. A MODEL CASCADE

The main parameters of the isotope separation cascade employed as the calculation model are listed in Table I. This is a tapered cascade, having a constant cut and separation factor over every stage. The feed is the natural uranium of 0.714% enrichment, and the product is 3.251% enriched uranium. The flow rate is normalized as the product flow becomes one. The nominal flow rate and enrichment distributions are illustrated in Fig. 8.

The number of centrifuges required amounts to as much as 800 000 for the model cascade in which the product of enriched uranium is 2000 ton/year.

In evaluations of the dynamic fluctuations, the continuous cascade model, shown in Sec. II, is employed. To assure the validity of this model, the steady-state solutions of Eqs. (20) and (22) were calculated. The results agreed almost perfectly with the original discrete cascade solutions shown in Fig. 8. The analytical forms of the steady-state solutions are presented in the Appendix.

TABLE I

Performance of a Model Cascade

Number of Stages	Rectifier Stripper	22 15
Cut Separation Gain	θ ϵ	0.4825[a] 0.07
Flow Rate[b]	Feed Product Waste	6.4425 1.0 5.4425
Enrichment (%)	Feed Product Waste	0.714 3.251 0.248

[a] $\theta = (1/2) - (\epsilon/4)$.
[b] Arbitrary units.

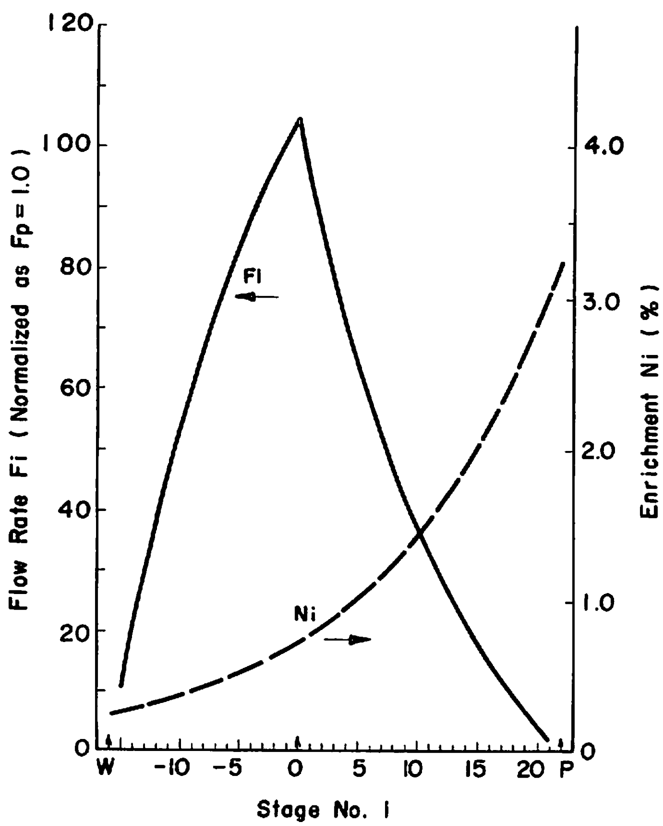

Fig. 8. Nominal flow rate and enrichment distributions in the cascade.

VI. STATIC DEVIATION OF FLOW RATE

Deviations of Cut and Separation Gain

Deviations of the cut and the separation gain are attributable mainly to manufacturing tolerance of the centrifuges, the aging effects which can be dealt with as static fluctuations, and the static control error of flow controllers. In this paper, the following two types of deviation are considered:

Uniform type: Standard deviations of the i'th-stage fluctuations are constant over the cascade.

Average type: Standard deviations of the i'th-stage fluctuations are inversely proportional to the square root of the i'th-stage feed flow rate.

The uniform type corresponds to the situation where performance of a stage is completely determined by a stage controller and the stage fluctuation is caused by an error of the controller, while the average type corresponds to the situation where there are no controllers or where a controller is provided for each centrifuge, not to the overall stage.

For the average type, some supplemental explanations are presented. When the i'th stage consists of K_i centrifuges and the cut deviation of the k'th centrifuge from the nominal value is $\delta\theta_{ik}$, the cut deviation of the i'th stage, $\delta\theta_i$, can be approximated by

$$\delta\theta_i = \frac{1}{K_i}\sum_{k=1}^{K_i} \delta\theta_{ik} \quad . \qquad (90)$$

If the standard deviation of $\delta\theta_{ik}$ is σ_θ, which is independent of i and k, the expectation of $\delta\theta_i^2$, $\sigma_{\theta_i}^2$, is expressed as

$$\sigma_{\theta_i}^2 = \sum_{k=1}^{K_i} \frac{\sigma_\theta^2}{K_i^2} = \frac{1}{K_i}\sigma_\theta^2 \quad . \qquad (91)$$

As the number K_i is proportional to the i'th-stage feed flow rate F_i, Eq. (91) can be written as

$$\sigma_{\theta_i} \propto \frac{1}{\sqrt{F_i}}\sigma_\theta \quad . \qquad (92)$$

Deviation of the separation gain can also be approximated by the form of Eq. (90); therefore, σ_{ϵ_i} is also inversely proportional to $\sqrt{F_i}$.

Numerical calculations are performed for the next six cases.

Case 1: σ_θ/θ = 1%, uniform type

Case 2: σ_θ/θ = 1%, average type

Case 3: σ_θ/θ = 1%, increases deterministically over the entire cascade

Case 4: σ_ϵ/ϵ = 5%, uniform type

Case 5: σ_ϵ/ϵ = 5%, average type

Case 6: σ_ϵ/ϵ = 5%, increases deterministically over the entire cascade.

In cases 2 and 5, the amounts of deviations are represented by the values at the 8'th stage. Cases 3 and 6 are deterministic cases which were added for comparison with the stochastic cases.

Deviations of Flow Rate and Enrichment

The deviations of flow rate and enrichment caused by the cut fluctuation (cases 1, 2, and 3) are presented in Figs. 9 and 10. The enrichment deviations of cases 4, 5, and 6 are shown in Fig. 11. The fluctuations of the product and waste uranium are summarized in Table II.

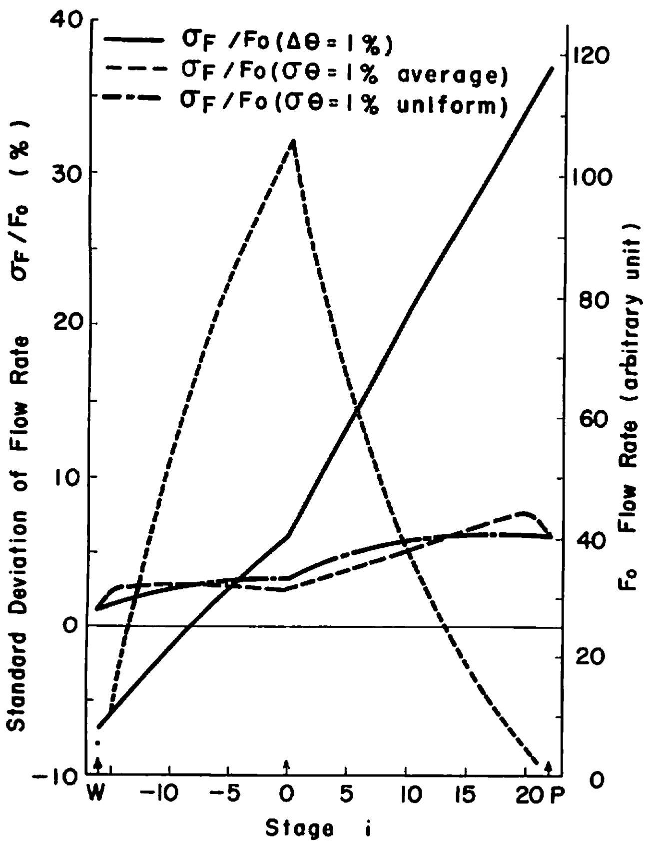

Fig. 9. Flow rate deviations caused by cut fluctuation.

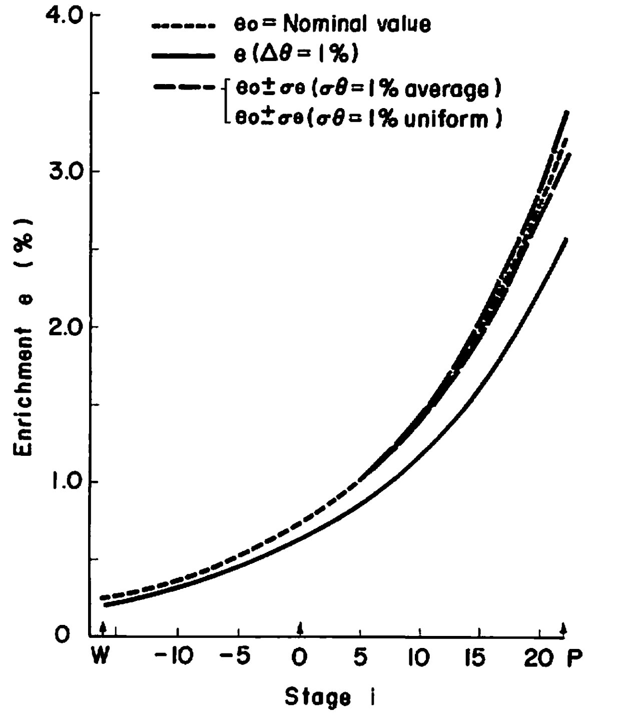

Fig. 10. Enrichment deviations caused by cut fluctuation.

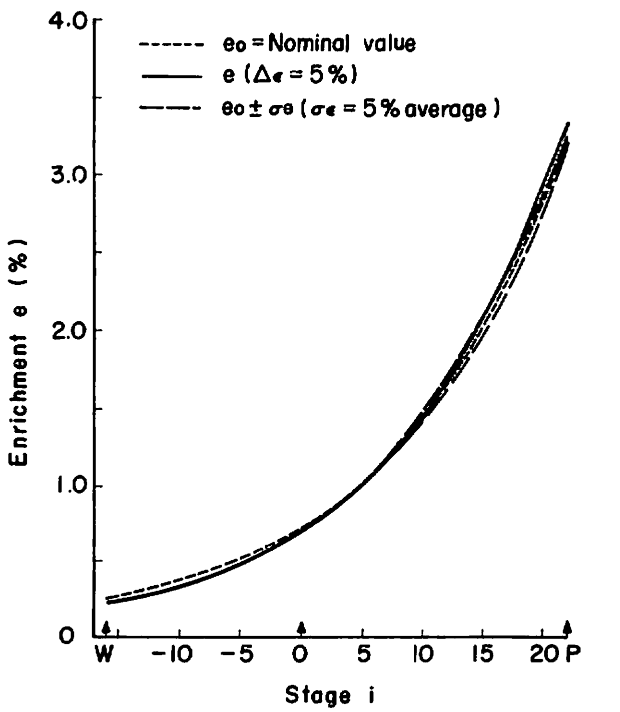

Fig. 11. Enrichment deviations caused by separation gain fluctuation.

TABLE II

Product and Waste Uranium Deviation of Flow Rate and Enrichment

Case	Product		Waste	
	Flow Rate $F_P \pm \sigma_{F_P}$	Enrichment $e_P \pm \sigma_{e_P}(\%)$	Flow Rate $F_W \pm \sigma_{F_W}$	Enrichment $e_W \pm \sigma_{e_W}(\%)$
1	1.0 ± 0.062	3.25 ∓ 0.14	5.443 ∓ 0.062	0.25 ∓ 0.01
2	1.0 ± 0.062	3.25 ∓ 0.15	5.443 ∓ 0.062	0.25 ∓ 0.01
3	1.373	2.60	5.069	0.20
4	1.0	3.25 ± 0.02	5.443	0.25 ∓ 0.005
5	1.0	3.25 ± 0.02	5.443	0.25 ∓ 0.004
6	1.0	3.36	5.443	0.23

Note: σ_{F_P} is the square root of the P'th diagonal element of the covariance matrix ϕ_{ff}. σ_{e_P}, σ_{F_P}, and σ_{e_W} are the same as σ_{F_P}. Flow rates are represented in the unit, such that the nominal product flow rate becomes one.

From these results, the following conclusions are obtained:

1. A cut fluctuation of 1% causes a product flow deviation of 6.2% ($3\sigma = 19\%$). In case 3 the product flow rate increases to 140% of the nominal rate; thus, the systematic shift of the cut over the cascade must be forbidden from a standpoint of flow control.

2. Deviation of the product enrichment is about 0.15% (relative 0.15/3.25 = 4.6%) in both cases 1 and 2.

3. A separation gain fluctuation of 5% causes a product enrichment deviation of 0.02% (relative 0.02/3.25 = 0.6%). The influence of this fluctuation is unexpectedly small compared with the effect of the cut fluctuation.

Strictly speaking, the separation gain of a centrifuge is influenced by the feed flow rate to the centrifuge, and when it works in the neighborhood of the point of maximum separation power, the relation between them is approximated by

$$\epsilon_i^2 F_i = \text{constant}$$

or

$$\frac{\delta\epsilon_i}{\epsilon_i} = -\frac{1}{2}\frac{\delta F_i}{F_i} \quad . \tag{93}$$

Consequently, the flow rate fluctuation, having interstage correlation, necessarily causes a deviation of the separation gain, having interstage correlation, and in this case, the enrichment fluctuation will be larger than the case where the i'th-stage separation gain deviation calculated by Eq. (93) is assumed to have no interstage correlations. However, the effect of actual separation gain deviation is small, and the above problem was considered not sufficiently important to be dealt with on a strict basis.

VII. THE DYNAMIC FLUCTUATION OF FLOW RATE

Fluctuation of Cut

In this section, flow and enrichment fluctuations caused by cut fluctuation, which are considered important in discussing plant control, are analyzed. The consequence of a separation factor fluctuation is rather small, as shown in the previous section; furthermore, the separation factor is not treated herein.

The causes of dynamic deviation of the cut are the aging effects, failure of the centrifuge, error of the controller, and other reasons. Consider the following two fluctuation types, as well as those in Sec. VI.

Uniform type: Deviation of the i'th-stage cut is the pulse series shown in Fig. 7 or Eqs. (83) and (84), where σ_i and $1/\beta_i$ are constants σ and $1/\beta$ over the cascade.

Average type: σ_i^2 and $1/\beta_i$ are inversely proportional to the i'th-stage feed flow rate.

These types correspond to the situations mentioned in Sec. VI, and the mean pulse width of the average type can be explained as follows. Assuming that the i'th separation stage consists of K_i centrifuges and the mean time width (in which the performance of a centrifuge is regarded as constant) is Δw, the mean time width of the i'th stage is $\Delta w/K_i$, and K_i is proportional to the i'th-stage feed flow rate F_i. Therefore, $1/\beta_i$ is inversely proportional to F_i.

Numerical calculations are presented for two cases:

Case 1: $\sigma_\theta/\theta = 1\%$, uniform type

Case 2: $\sigma_\theta/\theta = 1\%$, average type.

In each case, $1/\beta_i$ is changed parametrically. The deviation and the pulse width of the average type are represented by the values of the 8'th stage.

The Fluctuation of Flow Rate

Time constants of the cascade are presented in Table III, where the unit of time is the holdup time of a separation stage T. The largest one is about 250 T, and plant dynamics are characterized strongly by this constant.

First, the results of case 1 are presented. An example of the impulse response is illustrated in

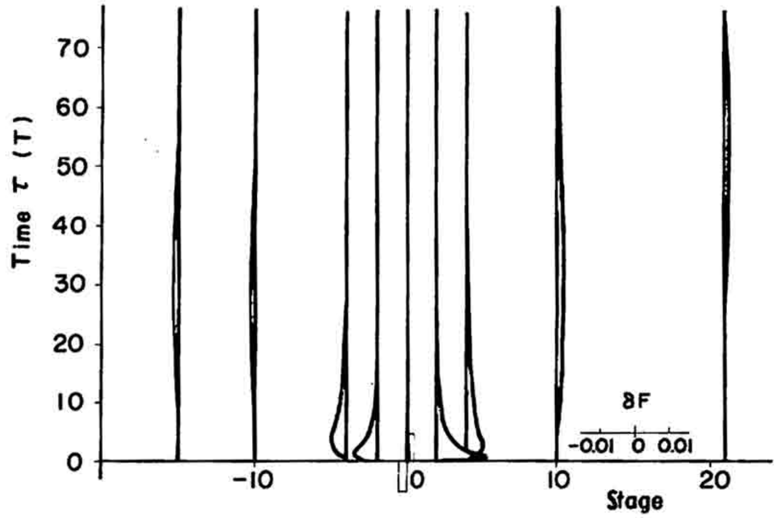

Fig. 12. Impulse response at selected stages due to a 1% cut change at the feed stage.

TABLE III

Time Constants of Modes

Mode No. n	Time Constant $1/\lambda_n$ [a]
1	248
2	70.1
3	31.8
4	18.1
5	11.6
.	
.	
.	
10	2.92
.	
.	
.	
15	1.30
.	
.	
.	
20	0.731
.	
.	
.	

[a]The unit is the holdup time of a separation stage.

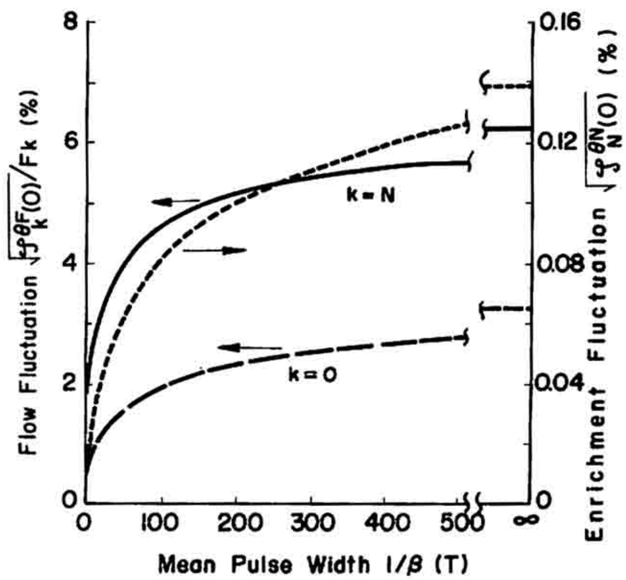

Fig. 13. Fluctuations of flow and enrichment as functions of mean pulse width (uniform type).

Fig. 12. Figure 13 shows the relations between $1/\beta$ and $\phi_k^{\theta F}(0)$ at the top and the feed stages ($k = N$ and 0). The top-stage enrichment fluctuation $\sqrt{\phi_N^{\theta N}(0)}$ as a function of $1/\beta$ is also illustrated in Fig. 13. The autocorrelation function $\phi_N^{\theta F}(\tau)$ as a function of time τ is presented in Fig. 14, where the top-stage nominal flow rate is normalized to 2.0725, as shown in Fig. 8.

From these results, the following conclusions are obtained:

1. For the infinite mean pulse width, the fluctuation coincides with the result of static evaluation.

2. If the mean pulse width is nearly equal to the holdup time T, fluctuations of the flow rate and the enrichment are rather small. As the width becomes a few tens of T, they become 3 and 0.04%, respectively, at the top stage. Therefore, if the cut of each stage can be controlled within the accuracy of 1% and duration of the deviation is a few tens of T, no significant problems exist concerning the flow control.

3. When the mean pulse width $1/\beta$ is larger than the time constant of the first mode $1/\lambda_1$, the autocorrelation function (shown in Fig. 14) decreases steeply at $\tau = 1/\beta$. However, for a smaller value of $1/\beta$, the decreasing characteristics of the first mode become significant.

Regarding case 2, fluctuations of flow and enrichment corresponding to those in Figs. 13 and

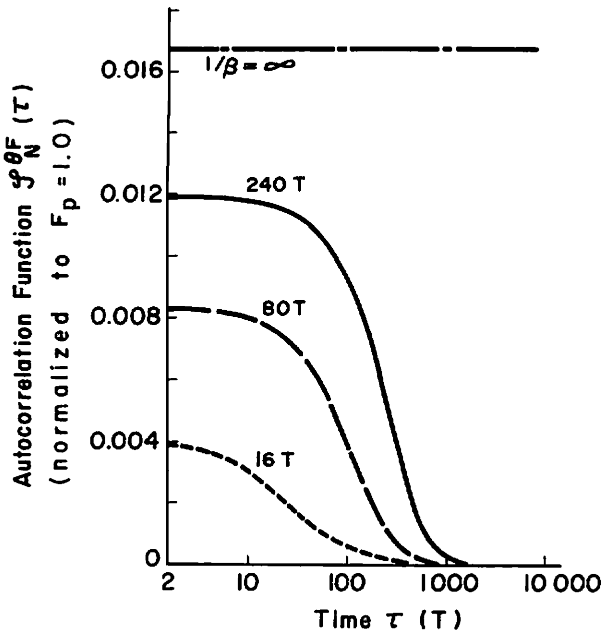

Fig. 14. Autocorrelation function $\varphi_N^{\theta F}$ as functions of time τ (uniform type).

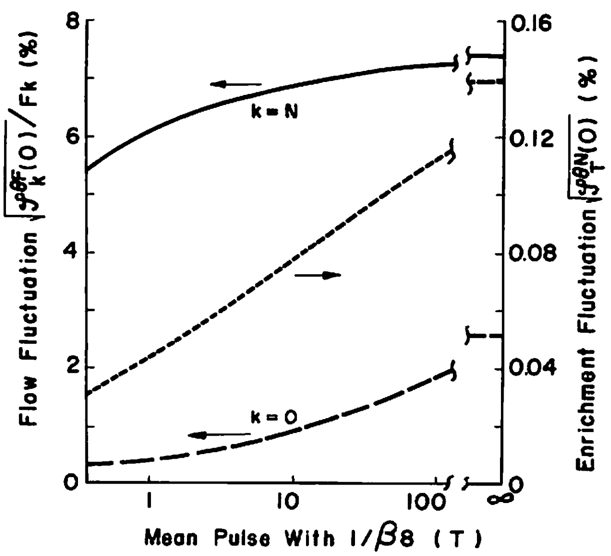

Fig. 15. Fluctuations of flow and enrichment as functions of mean pulse width (average type).

14 are shown in Figs. 15 and 16, and the following conclusions are obtained.

1. The mean pulse width at the 8'th stage $1/\beta_8$ = 100 T corresponds to $1/\beta_{21}$ = 2300 T and $1/\beta_0 = 45$ T. In this case, flow fluctuation at the top stage is almost saturated and nearly equal to that of $1/\beta_8 = \infty$. On the other hand, it is still increasing at the feed stage (k = 0), and the curve in Fig. 15 becomes convex toward the lower side of $1/\beta_8$.

2. Even the smallest stage (i.e., the top stage) consists of 900 centrifuges for the model cascade. Therefore, the cut fluctuation of a stage caused by a failure of one centrifuge is too small to cause a significant flow fluctuation.

3. Conclusion (3) of case 1 also applies in this case.

VIII. PERMISSIBLE FLUCTUATIONS OF STAGE PARAMETERS

Static and dynamic permissible fluctuations of stage parameters, to ensure a given cascade performance, are discussed in this section. The required performance is set as follows:

1. The stationary deviation of each stage flow rate from the nominal value should be $<10\%$ (3σ = 10%).

2. The stationary deviation of the product enrichment should be $<10\%$ ($3.25 \pm 3\sigma\%$, 3σ = 0.325%).

3. The dynamic fluctuation of each stage flow rate should be $<10\%$.

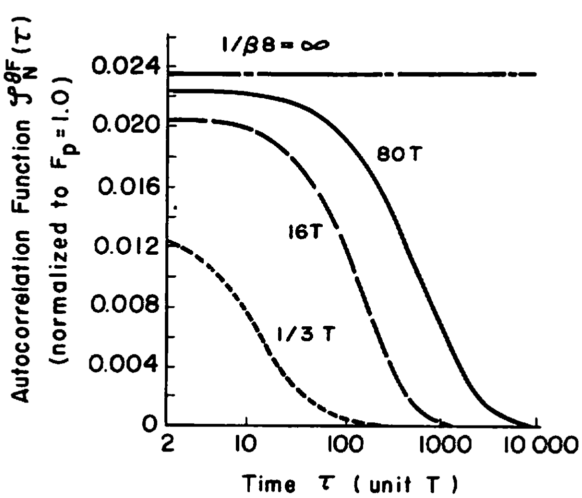

Fig. 16. Autocorrelation function $\varphi_N^{\theta F}$ as a function of time τ (average type).

Permissible fluctuations to satisfy requirements (1) and (2) are presented in Figs. 17a and and 17b. The stage where the flow rate deviation is the largest is the 18'th stage for the uniform type or the 20'th stage for the average type (see Fig. 9).

Observing these figures, it is evident that fluctuation of the separation gain can be rather large, but a high accuracy is required for the cut. Let it be assumed that the deviation of a stage cut is determined by a manufacturing error for each centrifuge; i.e.,

$$\sigma_{stage} = \left(\frac{1}{\text{number of centrifuges}}\right)^{1/2} \sigma_{cent} \quad .$$

The top stage, which is the smallest stage, consists of about 900 centrifuges, and as the deviation of the separation gain can easily be made less than several percent, the maximum permissible σ_{stage} is

$$\sigma_{stage} = \begin{cases} 0.5\% \text{ for the uniform type} \\ 2.0\% \text{ for the average type.} \end{cases} \qquad (94)$$

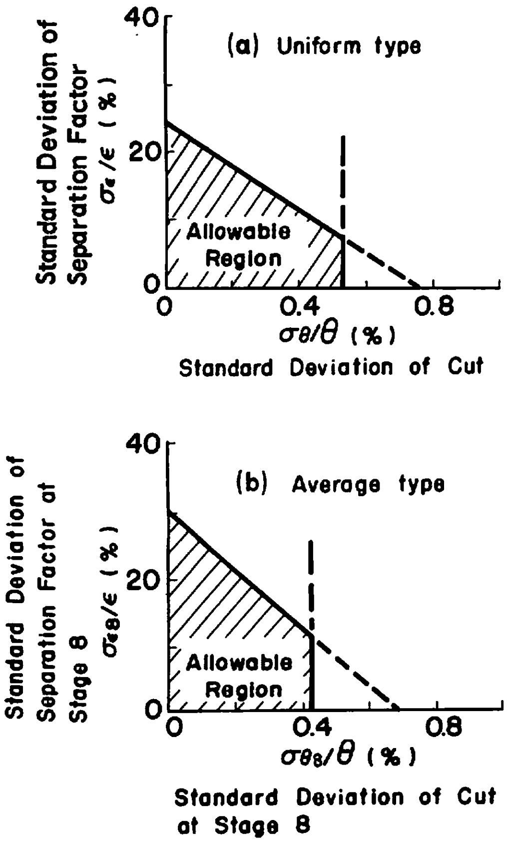

Fig. 17. Permissible regions of fluctuations in cut and separation factor to satisfy required stationary performance.

Therefore,

$$\sigma_{cent} = \begin{cases} 15\% \text{ for the uniform type} \\ 60\% \text{ for the average type.} \end{cases} \qquad (95)$$

These values are far outside the range where the linear approximation [Eq. (90)] is valid; however, it is correct to state that the permissible manufacturing error in the cut of each centrifuge can be rather large, and an error less than the above values can be easily realized. Therefore, while the cut of each centrifuge may deviate greatly, the mean value of the centrifuges averaged over a stage must be a nominal value with high accuracy; i.e., σ_{stage} of Eq. (94). If it is impossible to make the deviation of the mean value less than σ_{stage}, the cut must be controlled by the flow controller; in this case, Fig. 17 shows the permissible error in control.

For requirement (3), permissible dynamic fluctuations of the cut are given in Figs. 18a and 18b as functions of the mean pulse width. We have considered failure of the centrifuge, control error of the flow rate, and other reasons as causes of dynamic cut fluctuation. Centrifuge failure has no significant effect on the stage cut; consequently, the figures show required performance of the flow controller, if any.

IX. CONCLUSIONS

The uranium-enriching plant is a huge cascade composed of a vast number of centrifuges. Up to now, analysis of the system has been effected deterministically. The stochastic treatment is also necessary, especially to decide the tolerances for centrifuges and to define the performance of plant control systems.

First, the influence of stochastic deviation of stage performance was evaluated by using classic stochastic theory. Then, the time behavior caused by the random fluctuation of stage parameters was evaluated by the theory of stationary random processes.

The main conclusions obtained by numerical calculations are as follows:

1. If the cut of each stage is distributed at random with a standard deviation of 1%, the deviations of product flow and enrichment are about 6 and 0.15% (absolute), respectively.

2. If the stage separation gain fluctuates with σ = 5%, the product enrichment fluctuates by ±0.02% (absolute).

3. To limit the fluctuation of flow (all stages) and product enrichment within 10% (= 3σ), allowable tolerances of the cut and gain are as shown in Fig. 17.

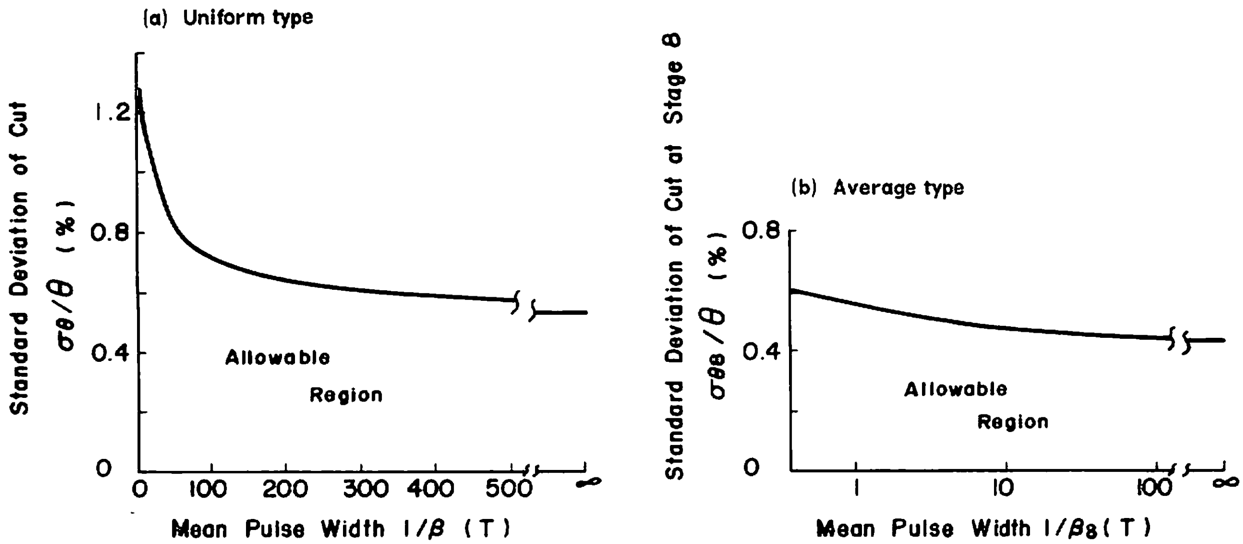

Fig. 18. Permissible dynamic fluctuation of cut to satisfy required dynamic performance.

4. There are many centrifuges in a stage, and the cut of a stage is the mean value of those centrifuges. Therefore, the relative deviation of the cut of a stage is sufficiently small. However, the mean value does not necessarily lie within the specified tolerance. The control system must correct the mean value to a nominal one.

5. When the perturbations are time dependent, the magnitude of the response increases monotonically with the time interval of perturbation. With an infinitely long time interval compared with the system time constant, the result merges into the steady problem described above. The time constant of the system is about 250 times the holdup time of a centrifuge for the model cascade.

6. If the fluctuation of the cut from the nominal value is approximated by the rectangular pulse series with an average width of a few tens of the holdup time of a centrifuge and with an average height of 1%, the fluctuation of product flow becomes about 3%.

7. The latitude of cut fluctuation as a function of mean time interval of the disturbance is illustrated in Fig. 18.

8. The cut deviation affects product enrichment more through direct mass balance (mixing) than through changes in the separation gain.

Thus, the relations have been analyzed between the fluctuations of cut and gain of centrifuges and the system performance. The numerical illustrations as well as the formulas should be useful in the design and operation of a centrifuge plant.

APPENDIX

The steady-state solutions of Eqs. (20) and (22) can be obtained analytically:

$$F(X) = \begin{cases} \dfrac{2F_f \exp[\epsilon(M+1)]\{\exp[\epsilon(N+1)]-1\}}{\epsilon\{\exp[\epsilon(M+N+2)]-1\}} \times [1-\exp(-\epsilon X)] , \\ \qquad 0 \le X \le M+1 \\ \dfrac{2F_f \exp[\epsilon(M+1)]\{1-\exp[-\epsilon(M+1)]\}}{\epsilon\{\exp[\epsilon(M+N+2)]-1\}} \times \{\exp[-\epsilon(X-M-N-2)]-1\} , \\ \qquad M+1 \le X \le M+N+2 \end{cases} \tag{A.1}$$

$$G(X) = \begin{cases} \dfrac{2F_f N_f \exp[-\epsilon(M+1)]\{1-\exp[-\epsilon(N+1)]\}}{\epsilon\{1-\exp[-\epsilon(M+N+2)]\}} \times [\exp(\epsilon X)-1] , \\ \qquad 0 \le X \le M+1 \\ \dfrac{2F_f N_f \exp[-\epsilon(M+1)]\{\exp[\epsilon(M+1)]-1\}}{\epsilon\{1-\exp[-\epsilon(M+N+2)]\}} \times \{1-\exp[\epsilon(X-M-N-2)]\} , \\ \qquad M+1 \le X \le M+N+2 \end{cases} \tag{A.2}$$

From Eqs. (A.1) and (A.2), the enrichment is given by

$$N(X) = N_f \exp[\epsilon(X - M - 1)] \quad . \qquad (A.3)$$

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