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Sensitivity Analysis of Ideal Centrifuge Cascade for Producing Slightly Enriched Uranium

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Characteristics of an ideal cascade are analyzed by two differential equations representing the conservation of UF_6 and $^{235}UF_6$ flow. The controlling variables are identified as the cut and the separation factor of centrifuges and of stages as well as feed flow rate. The controlled variables are flow rate and enrichment of stages, especially of the product and waste. The sensitivity of the controlled variables to the controlling variables are analyzed by linearizing the conservation equations, and analytic expressions are obtained. The change in the separative work of the cascade is a sum of changes in the separative work of the constituent centrifuges. When the flow rate is chosen to optimize the separative work of a single centrifuge, the plant separative work is maximum and stationary at the rated feed flow. It has been demonstrated in a few examples that these simple relations for the ideal cascade are useful for the planning, design, and operation of cascade plants.

I. INTRODUCTION

The work reported here is an investigation of a uranium enriching cascade using centrifuges.

The theory of the cascade for isotope separation is well established.[1] Solutions for square, tapered and ideal cascades have been investigated for static and dynamic situations.[2-9] These investigations deal with the cascade which has a unique set of deterministic system parameters.

It is known that the steady state plant performance of the cascade is governed by the cut and the separation factor of stages as well as the feed flow to the cascade. These parameters are not always identical with the design values. For example, the cut may fluctuate within the error band of the stage flow controller. The cut and the separation factor of centrifuges will distribute with certain deviations caused by manufacturing tolerances. The shut down of a centrifuge reduces the stage separative factor. Furthermore, feed flow rate may be controlled in order to meet the varying demand of product enrichment.[10]

It is therefore most important to investigate

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[9] K. HIGASHI, A. OYA, and J. OISHI, *Nucl. Sci. Eng.*, **32**, 159 (1968).

[10] T. KIGUCHI, H. MOTODA, and T. KAWAI, "Stochastic Fluctuation in Uranium Enriching Cascade by Centrifuge Process," submitted to the *Nuclear Technology*.

the effect of such variations on the plant performances for the design and operation of the cascade.

The ideal cascade is considered to be practical for enriching plants composed of centrifuges. In the present paper, the authors derive in analytic forms the sensitivity of parameters on the performances of the ideal cascade. The results are then applied to some operational problems.

II. SYSTEM EQUATIONS

Flow Relations

Let F_i and G_i be the total UF_6 and $^{235}UF_6$ flow rates into the i'th stage. Primes ′ and ″ denote the enriched and depleted flow out of the stage, as shown in Fig. 1. From the conservation of mass flow the following equations are derived:

$$F_i = F'_{i-1} + F''_{i+1} + \delta_{i,I} F_f \tag{1a}$$

$$G_i = G'_{i-1} + G''_{i+1} + \delta_{i,I} N_f F_f \tag{1b}$$

$$\delta_{i,I} = 1\ (i = I) \quad , \quad = 0\ (i \neq I) \quad ,$$

where N denotes enrichment and f (subscript) denotes feed stage. The stage number is defined in Fig. 2. For the top and the bottom stage $i = L - 1$ and 1, respectively,

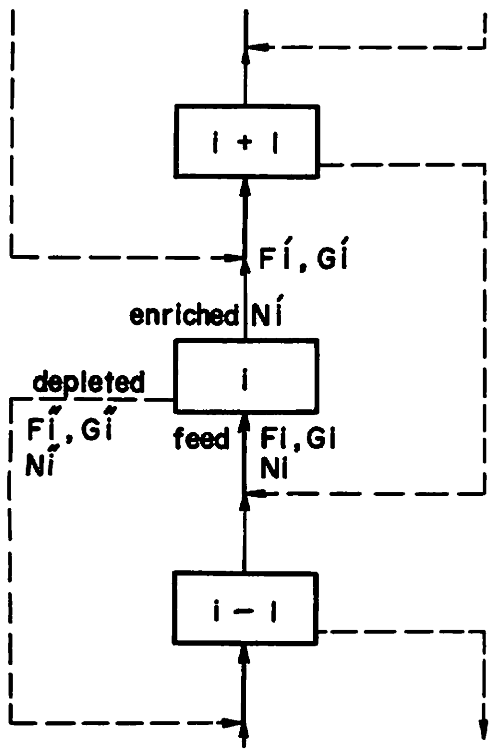

Fig. 1. Interstage flow rates at the i'th stage of the cascade.

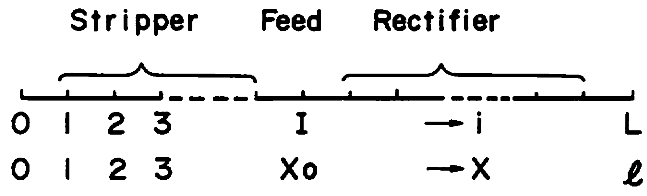

Fig. 2. Definition of stage number i and variable x.

$$F_{L-1} = F'_{L-2} \quad , \quad F_1 = F''_2 \tag{2a}$$

$$G_{L-1} = G'_{L-2} \quad , \quad G_1 = G''_2 \quad , \tag{2b}$$

which correspond to Eqs. (1a) and (1b) if the flow rates are zero at the fictious stages 0 and L.

$$F''_L = F'_0 = 0 \tag{3a}$$

$$G''_L = G'_0 = 0 \quad . \tag{3b}$$

Therefore, Eqs. (1a) and (1b) hold for all $i \in [1, L-1]$ with the boundary conditions (3a) and (3b). The enrichment N_i is obtained by

$$N_i = G_i / F_i \quad . \tag{4}$$

Cut and the Separation Gain

The cut θ_i, separation factor $(\alpha\beta)_i$, and separative gain ϵ_i are defined by

$$F'_i \equiv \theta_i F_i \quad , \quad F''_i \equiv \bar{\theta}_i F_i \quad , \quad \bar{\theta}_i \equiv 1 - \theta_i \tag{5}$$

$$\frac{N'_i}{1 - N'_i} \equiv (\alpha\beta)_i \frac{N''_i}{1 - N''_i} \tag{6}$$

$$(\alpha\beta)_i \equiv 1 + 2\epsilon_i \quad . \tag{7}$$

Assumptions and Approximations

The gain ϵ_i is very small compared with 1, and so the following approximations hold:

$$(N'_i - N_i)/N_i \ll 1 \quad , \quad (N_i - N''_i)/N_i \ll 1 \quad . \tag{8}$$

Further it is assumed that

$$N_i \ll 1 \quad . \tag{9}$$

Cut for the Ideal Cascade

The condition for the ideal cascade[1]

$$\theta = \frac{1 - N + \sqrt{\alpha\beta}\, N}{1 + \alpha} \quad , \quad \alpha = \beta \tag{10}$$

reduces to

$$\theta \fallingdotseq \frac{1 - N + (1 + \epsilon) N}{1 + (1 + \epsilon)} \fallingdotseq \frac{1}{2} - \frac{1}{4}\epsilon \quad , \tag{11}$$

with approximations (8) and (9). It is assumed that the stages are made of identical centrifuges. Therefore, in the ideal cascade, the stage gain ϵ is constant and independent of stage number, as is the stage cut θ. These constant values will be denoted by ϵ^0 and θ^0.

Fundamental Equations

Inserting the definition of cut, Eq. (5), into Eq. (1a), the interstage relation of flow rates follows:

$$-\tfrac{1}{2}(F_{i-1} - 2F_i + F_{i+1}) - \tfrac{1}{2}(F_{i+1} - F_{i-1}) + \theta_{i+1} F_{i+1} - \theta_{i-1} F_{i-1} = \delta_{i,l} F_f \quad . \qquad (12a)$$

In order to derive the corresponding relation for G, preliminary considerations are necessary.

It is shown in Appendix A that the concentrations of the enriched and depleted flow are

$$\begin{aligned} N_i' &= (1 + 2\bar{\theta}_i \epsilon_i) N_i \\ N_i'' &= (1 - 2\theta_i \epsilon_i) N_i \quad . \end{aligned} \qquad (13)$$

Substituting Eqs. (13), (4), and (5) into Eq. (1b), we finally obtain

$$-\tfrac{1}{2}(G_{i-1} - 2G_i + G_{i+1}) - \tfrac{1}{2}(G_{i+1} - G_{i-1}) + \theta_{i+1}G_{i+1} - \theta_{i-1}G_{i-1} + 2(\epsilon\theta\bar{\theta}G)_{i+1} - 2(\epsilon\theta\bar{\theta}G)_{i-1} = \delta_{i,l} N_f F_f \quad . \qquad (12b)$$

The stagewise Eqs. (12a) and (12b) are approximately described by the continuous stage variable x (Fig. 2) as

$$-\frac{1}{2}\frac{d^2}{dx^2}F - \frac{d}{dx}\{(1 - 2\theta)F\} = \delta(x - x_0)F_f \qquad (14a)$$

$$-\frac{1}{2}\frac{d^2}{dx^2}G + \frac{d}{dx}\{(-1 + 2\theta + 4\epsilon\theta\bar{\theta})G\} = \delta(x - x_0)N_f F_f \quad . \qquad (14b)$$

The approximation is based on the assumption that the quantities vary slowly with respect to the stage. When the relation for the ideal cascade, Eq. (11), is inserted, Eqs. (14a) and (14b) are further reduced as

$$-\frac{1}{2}\frac{d^2}{dx^2}F - \frac{1}{2}\epsilon^0\frac{d}{dx}F = \delta(x - x_0)F_f \qquad (15a)$$

$$-\frac{1}{2}\frac{d^2}{dx^2}G + \frac{1}{2}\epsilon^0\frac{d}{dx}G = \delta(x - x_0)N_f F_f \quad . \qquad (15b)$$

Reference Solutions

The solution of Eqs. (15a) and (15b) satisfying the boundary condition $F = G = 0$ at $x = 0,\ l$ and continuous at $x = x_0$ can be readily obtained (see Appendix B):

$$\left.\begin{aligned} F &= C\{\exp[\epsilon^0(l - x_0)] - 1\}[1 - \exp(-\epsilon^0 x)] \qquad 0 \le x \le x_0 \\ F &= C\{\exp[\epsilon^0(l - x)] - 1\}[1 - \exp(-\epsilon^0 x_0)] \qquad x_0 \le x \le l \\ C &= \frac{2F_f \exp(\epsilon^0 x_0)}{\epsilon^0[\exp(\epsilon^0 l) - 1]} \quad ; \end{aligned}\right\} \qquad (16a)$$

$$\left.\begin{aligned} G &= C'\{1 - \exp[-\epsilon^0(l - x_0)]\}[\exp(\epsilon^0 x) - 1] \qquad 0 \le x \le x_0 \\ G &= C'\{1 - \exp[-\epsilon^0(l - x)]\}[\exp(\epsilon^0 x_0) - 1] \qquad x_0 \le x \le l \\ C' &= \frac{2N_f F_f \exp(-\epsilon^0 x_0)}{\epsilon^0[1 - \exp(-\epsilon^0 l)]} \quad . \end{aligned}\right\} \qquad (16b)$$

From Eqs. (16a), (16b), and (4) follows the equation

$$N = N_f \exp[\epsilon^0(x - x_0)] \quad . \qquad (17)$$

For later references, the outlet quantities are represented below, as special cases of Eq. (16):

$$\left.\begin{aligned} F_w &= \theta\ F_1 = F_f \frac{N_P - N_f}{N_P - N_w} \\ F_P &= \theta\ F_{l-1} = F_f \frac{N_f - N_w}{N_P - N_w} \quad , \end{aligned}\right\} \qquad (18)$$

where subscripts P and w denote product and waste respectively. Obviously they satisfy the conservation laws

$$F_P + F_w = F_f \qquad (19a)$$

$$N_P F_P + N_w F_w = N_f F_f \quad . \qquad (19b)$$

Stage Parameters and Centrifuge Parameters

The cut and the separative gain of a stage is composed of the cut and the separative gain of the constituent centrifuges. It is shown in Appendix C that the variation of the cut θ_i and the separative gain ϵ_i of a stage i is the average of the variations of the cut θ_{ij} and the separative gain ϵ_{ij} of j'th centrifuges belonging to the stage i if their variations are small,

$$\delta\epsilon_i = \left(\sum_{j=1}^{i} \delta\epsilon_{ij}\right)\Big/ J_i \qquad (20)$$

$$\delta\theta_i = \left(\sum_{j=1}^{i} \delta\theta_{ij}\right)\Big/ J_i \quad , \qquad (21)$$

where J_i, the number of centrifuges in a stage, is designed to be proportional to the stage flow:

$$J_i = F_i/f \quad , \qquad (22)$$

where f is the flow rate per centrifuge.

Flow and the Gain

The separative gain ϵ is a function of the flow rate per centrifuge. Consider the separative work of a centrifuge:

$$u = \tfrac{1}{2}\epsilon^2 f \quad . \qquad (23)$$

The operating condition will be set so that this function is maximum with respect to the flow. Thus the flow dependence of the separative gain is

$$\delta\epsilon/\epsilon^0 = -\delta f/(2f) \quad . \qquad (24)$$

III. SENSITIVITY ANALYSIS

Controlling Variables and Objective Functions

Equations (14a) and (14b) show how the flows F and G are related to θ, ϵ and F_f. The relation is shown schematically in Fig. 3. Feed flow and stagewise cut are the controlling variables.

The first order effects of system parameters θ_i, ϵ_i, and F_f on the flow distributions $F(x)$, $G(x)$, and hence $N(x)$ can be analyzed by linearization. These effects are additive, and we treat them separately.

The objective function may be either the separative work of the plant

$$U = F_P V(N_P) + F_w V(N_w) - F_f V(N_f) \tag{25}$$

or the amount of net sales

$$S = F_P P(N_P) + F_w P(N_w) - F_f P(N_f) \quad , \tag{26}$$

where V is the value function

$$V(N) = (2N - 1) \log \frac{N}{1 - N} \fallingdotseq - \log N \quad , \tag{27}$$

and P is the unit price of the uranium. The differential of U is obviously

$$\delta U = -F_P \frac{\delta N_P}{N_P} - F_w \frac{\delta N_w}{N_w} + F_f \frac{\delta N_f}{N_f} - \log \frac{N_P}{N_f} \delta F_P - \log \frac{N_w}{N_f} \delta F_w \quad . \tag{28}$$

The Effect of Separation Gain

As seen from Fig. 3, the change in separative gain $\delta\epsilon_i$ affects the $^{235}UF_6$ flow, G, and hence the enrichment, leaving the total flow F unchanged. By linearizing Eq. (14b), the following equation for δG is obtained:

$$- \frac{d^2}{dx^2} \delta G + \epsilon^0 \frac{d}{dx} \delta G = - \frac{d}{dx} \{2G_0(\delta\epsilon + 2\delta\theta)\} \quad . \tag{29}$$

The solution for the i'th stage change $\delta\epsilon_i$ is

$$\left.\begin{aligned} &\delta G = -C_i[\exp(\epsilon^0 x) - 1] && 0 \le x < x_i \\ &\delta G = C_i\{1 - \exp[-\epsilon^0(l - x)]\} && x_i < x \le l \\ &C_i = 2 \exp(-\epsilon^0 x_i) G(x_i) \delta\epsilon_i / [1 - \exp(-\epsilon^0 l)] \quad . \end{aligned}\right\} \tag{30}$$

The x-dependence of δG is similar to that of G in the region $x > \max(x_i, x_0)$ and $x < \min(x_i, x_0)$, so that $\delta N/N$ does not depend on the stage for some ranges of stages

$$\left.\begin{aligned} &\frac{\delta N(x)}{N(x)} = \frac{\epsilon^0 F_i \delta\epsilon_i}{F_f[\exp(\epsilon^0 x_0) - 1]} && x > \max(x_0, x_i) \\ &\frac{\delta N(x)}{N(x)} = - \frac{\epsilon^0 F_i \delta\epsilon_i}{F_f\{1 - \exp[-\epsilon^0(l - x_0)]\}} && x < \min(x_0, x_i) \end{aligned}\right\} \tag{31}$$

from which the change in the product and waste enrichment is obtained. By the use of equations (17), (20), and (22), these equations yield

$$\left.\begin{aligned} &\delta N_P = \frac{N_w N_P}{N_f - N_w} \frac{f}{F_f} \epsilon^0 \delta\epsilon_{ij} \\ &\delta N_w = - \frac{N_w N_P}{N_P - N_f} \frac{f}{F_f} \epsilon^0 \delta\epsilon_{ij} \quad . \end{aligned}\right\} \tag{32}$$

This result shows that the effect of the separative gain of a single centrifuge is independent of the stage in which the centrifuge is located.

The effect on the separative work is obtained by inserting Eq. (32) into Eq. (28). The result is simply

$$\delta U = \sum_{ij} f \epsilon^0 \delta\epsilon_{ij} = \sum_{ij} \delta(\tfrac{1}{2} f \epsilon_{ij}^2) \quad . \tag{33}$$

This is the sum of the change in the separative work of constituent centrifuges, leading to the

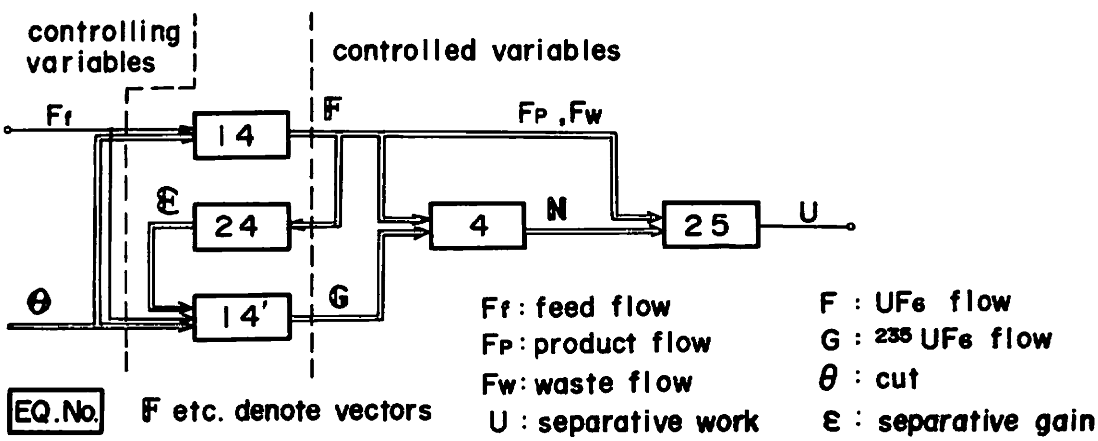

Fig. 3. Relations between controlling and controlled variables.

conclusion that the mixing loss is negligible in the first order of $\delta\epsilon$ in an ideal cascade.

Effect of the Feed Flow

Total flow F_i is proportional to the feed flow as readily seen from Eq. (16a):

$$\delta F(x)/F(x) = \delta F_f/F_f \equiv \Delta \quad . \tag{34}$$

This causes uniform changes in the separative gain $\delta\epsilon/\epsilon^0 = -\Delta/2$. Equation (14b) then becomes

$$-\frac{1}{2}\frac{d^2}{dx^2}G + \frac{d}{dx}\left\{\left(\frac{1}{2}\epsilon^0 + \delta\epsilon\right)G\right\} = \delta(x - x_0)N_f F_f(1 + \Delta) \quad , \tag{35}$$

and the solution is analogous to Eq. (16b), with the replacements

$$\epsilon_0 \rightarrow \epsilon_0(1 - \Delta) \quad , \quad F_f \rightarrow F_f(1 + \Delta) \quad .$$

Assuming $\Delta << 1$, the change δG can be obtained by differentiating the exact solution,

$$\left.\begin{aligned}
\frac{\delta G(x)}{G(x)} &= -\Delta\left\{-\epsilon^0 x_0 - 1 - \frac{\epsilon^0 l}{\exp(\epsilon^0 l) - 1} + \frac{\epsilon^0(l - x_0)}{\exp[\epsilon^0(l - x_0)] - 1} + \frac{\epsilon^0 x}{1 - \exp(-\epsilon^0 x)}\right\} + \Delta \ , \quad x < x_0 \\
\frac{\delta G(x)}{G(x)} &= -\Delta\left\{-\epsilon^0 x_0 - 1 - \frac{\epsilon^0 l}{\exp(\epsilon^0 l) - 1} + \frac{\epsilon^0(l - x)}{\exp[\epsilon^0(l - x)] - 1} + \frac{\epsilon^0 x_0}{1 - \exp(-\epsilon^0 x_0)}\right\} + \Delta \ , \quad x > x_0 \ .
\end{aligned}\right\} \tag{36}$$

Change in $\delta N/N$ is obtained by subtracting Δ from the above equations, especially at the top and bottom,

$$\left.\begin{aligned}
\left.\frac{\delta N}{N}\right)_{x=1} &= \Delta\left(\epsilon^0 x_0 + \frac{\epsilon^0 l}{\exp(\epsilon^0 l) - 1} - \frac{\epsilon^0(l - x_0)}{\exp[\epsilon^0(l - x_0)] - 1}\right) \\
\left.\frac{\delta N}{N}\right)_{x=l-1} &= \Delta\left(\frac{\epsilon^0 l}{\exp(\epsilon^0 l) - 1} - \frac{\epsilon^0 x_0}{\exp(\epsilon^0 l) - 1}\right) \ .
\end{aligned}\right\} \tag{37}$$

The result looks complicated, but it is transformed into simpler equations below (see Appendix D),

$$\left.\begin{aligned}
\left.\frac{\delta N}{N}\right)_w &= \frac{N_P}{N_P - N_f}\frac{U_T}{F_f}\Delta \\
\left.\frac{\delta N}{N}\right)_P &= -\frac{N_w}{N_f - N_w}\frac{U_T}{F_f}\Delta \ ,
\end{aligned}\right\} \tag{38}$$

where U_T is the total work of the centrifuges. This equation can also be simply derived from the previous result, Eq. (32), when it is noted that the feed flow causes $\delta\epsilon_{ij} = -\frac{1}{2}\epsilon^0\Delta$ for all centrifuges.

The change in the plant separative work by the feed flow is obtained by δF_P, δF_w, and δF_f from Eq. (34), δN_P and δN_w from Eq. (38),

$$\begin{aligned}
\delta U &= -\Delta \cdot U + \Delta\left[\frac{1}{2}(\epsilon^0)^2 f J_T\right] \times\left\{\frac{1}{F_f}\left(\frac{F_P N_w}{N_f - N_w} - \frac{F_w N_P}{N_P - N_f}\right)\right\} \\
&= -\Delta \cdot U + \Delta \cdot (U)\{1\} = 0 \quad ,
\end{aligned} \tag{39}$$

where Eqs. (18) and (D.1) have been used. This shows that the ideal cascade is operated at the optimum stationary condition when the rated flow is set by the condition (24).

Effect of the Cut

The change in F due to the cut change is obtained from Eq. (14a).

$$-\frac{1}{2}\frac{d^2}{dx^2}\delta F - \frac{\epsilon}{2}\frac{d}{dx}\delta F - 2\frac{d}{dx}(\delta\theta \cdot F) = 0 \quad . \tag{40}$$

This is the same form as Eq. (29) if we replace ϵ^0 in Eq. (29) by $-\epsilon^0$, and $-2G_0\delta\epsilon$ by $4\delta\theta_i F$. The solution is

$$\left.\begin{aligned}
\delta F &= C_i'[1 - \exp(-\epsilon^0 x)] & x \le x_i \\
\delta F &= -C_i'\{\exp[\epsilon^0(l - x)] - 1\} & x_i < x \\
C_i' &= -4\epsilon^0 x_0 F_i \delta\theta_i/[\exp(\epsilon_0 l) - 1] \ .
\end{aligned}\right\} \tag{41}$$

The change in the product flow due to $\delta\theta_i$ is expressed, using Eq. (17),

$$\delta F_P = \theta\delta F_{l-1} = \frac{1}{N_P - N_w}2\epsilon^0 G_i\delta\theta_i \quad . \tag{42}$$

The conservation of flow $\delta F_P + \delta F_w$ at the outlet is verified. The relative change of flow $\delta F(x)/F(x)$ is constant outside the region $\min(x_0, x_i) \le x \le \max(x_0, x_i)$, but varies inside, so that the flow per centrifuge is not constant through the stages, hence the separative gain. It seems to be difficult to solve δG analytically for the cut change (Appendix E). However, the product and waste enrichment will be analytically obtained from the consideration of separative work. We postulate that the separative work does not change by the small change in the cut in an ideal cascade. Thus using the three conservation equations (19a), (19b), and (28) for UF_6, $^{235}UF_6$, and U, we can solve δF_w, δN_P, δN_w in terms of δF_P. The results are

$$\left.\begin{aligned}
\delta F_w &= -\delta F_P \\
\frac{\delta N_P}{N_P} &= -\left(1 + \frac{N_w}{N_P - N_w}\log\frac{N_P}{N_w}\right)\frac{\delta F_P}{F_P} \\
\frac{\delta N_w}{N_w} &= \left(1 + \frac{N_P}{N_P - N_w}\log\frac{N_P}{N_w}\right)\frac{\delta F_P}{F_w} \ .
\end{aligned}\right\} \tag{43}$$

Thus the effect of cut on the plant output quantities is obtained in spite of the complexity of the flow change throughout the stages.

IV. ILLUSTRATIONS

The Model Plant

Table I shows the model plant which is chosen as an illustrative purpose. The flow F and enrichment are shown as functions of stage in Fig. 4. Also in this figure are plotted by dotted lines the corresponding quantities calculated by exact difference equations for an exactly ideal cascade [cut determined by Eq. (11) without the approximation].

TABLE I

Performance of a Model Cascade

Cut[a]	θ	0.4825
Separative Gain	ϵ	0.07
Number of Stages	Rectifier	22
	Stripper	15
Flow Rate[b]	Feed	6.4425
	Product	1.0
	Waste	5.4425
Enrichment (%)	Feed	0.714
	Product	3.251
	Waste	0.248
Separative Work[b]	U	4.239

[a] $\theta = \frac{1}{2} - \frac{1}{4}\epsilon$.

[b] Product flow unit.

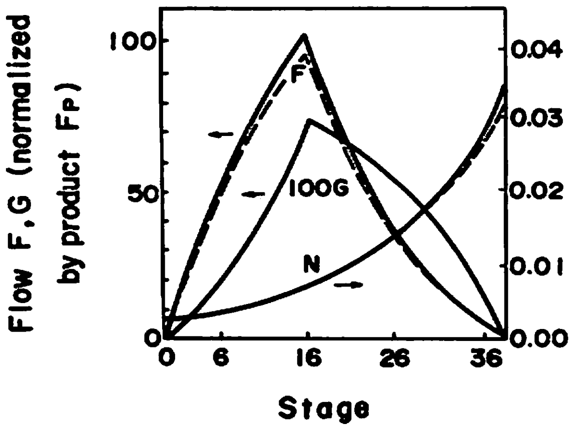

Fig. 4. Total flow, $^{235}UF_6$ flow and enrichment of a reference plant.

Illustrations of the Sensitivity

The calculated results of $\delta N_i/N_i$ by changes in the feed flow are illustrated in Fig. 5. Effects of cut change $\delta\theta_i\,(i = 6 \sim 31)$ on the flow are illustrated in Fig. 6. Figure 7 shows the effect of the separation gain on the enrichment. Other trivail relations ($\delta F_f \rightarrow \delta F_i$, $\delta\epsilon_i \rightarrow \delta F_i$) are omitted.

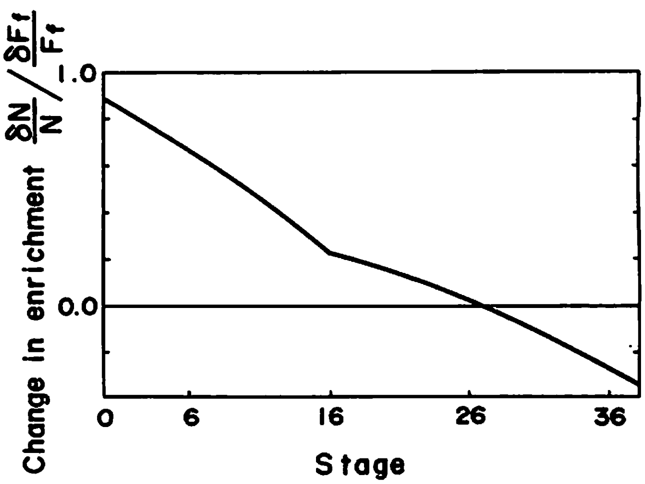

Fig. 5. Change in the enrichment due to feed flow variation.

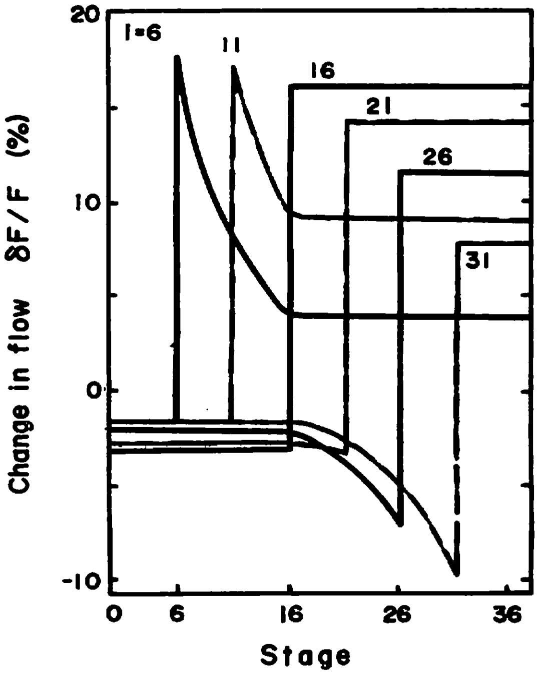

Fig. 6. Change in the flow due to cut variation at the i'th stage. ($\delta\theta = +0.1\ \theta°$).

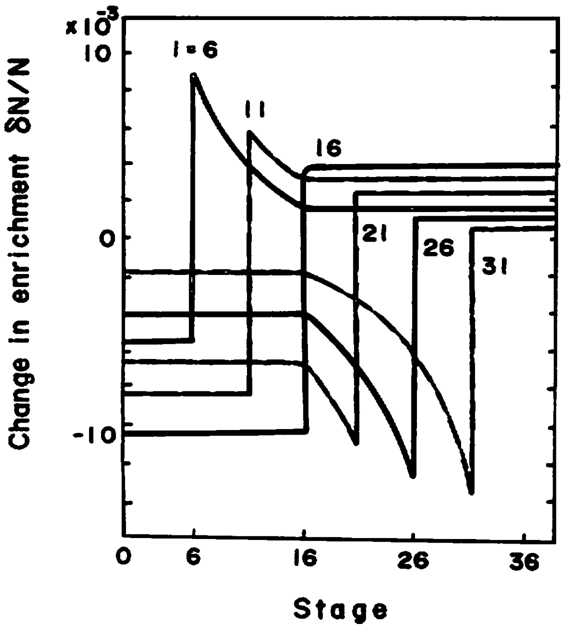

Fig. 7. Change in the enrichment due to separative gain variation at the i'th stage. ($\delta\epsilon = +0.1\ \epsilon^\circ$).

Optimum Operation for Varying Demand Enrichment

The consideration in the previous chapter showed that the separative work of an ideal cascade is kept invariant when feed flow and cut of the stages are varied. Cut of the stages may be adjusted before and during the operation, but not for controlling purposes, because there are too many cut to be controlled in the ideal cascade. The feed flow, on the other hand, is much simpler to control. Its effect on the flow and enrichment is also much simpler. It can meet the varying enrichment demand without mixing loss, while other methods such as outside blending, extraction at the mid-stage, or partial shut down of the centrifuges necessarily lose the separative value.

A comparison among the possible methods of operation is shown in Fig. 8. The relative change in the product enrichment $(\delta N/N)_P$ due to the relative change in the feed flow Δ is known to be -0.344 for the model plant.

Criterion for Replacement

According to calculated results, the response of the separative work of the plant δU to the change of separative gain of one element (one centrifuge or one set of centrifuges) $\delta\epsilon$ is well approximated by the following equation[11]:

$$\frac{\delta U}{U} = \frac{2}{N} \cdot \frac{\delta\epsilon}{\epsilon^0} \quad , \tag{44}$$

where U, N, and ϵ^0 denote the separative work of the plant, number of the elements, and nominal separative gain of the elements, respectively.

The economical loss of the plant δY is represented by

$$\frac{\delta Y}{Y} = -\frac{\delta U}{U} = -\frac{1}{N} \cdot \frac{2\delta\epsilon}{\epsilon^0} \quad , \tag{45}$$

where Y is the total cost for the separation work.

The deteriorated element with the separative gain $\epsilon^0 + \delta\epsilon$ should be replaced with a new one when

$$z < \delta Y \quad , \tag{46}$$

where z denotes the capital cost (depreciation account) of the element.

From Eqs. (45) and (46), the criterion for the replacement is given by

$$\frac{\epsilon^0 + \delta\epsilon}{\epsilon^0} < 1 - \frac{1}{2} \cdot \frac{Z}{Y} \quad , \tag{47}$$

where Z is the whole capital cost of the elements $(= N_Z)$. Figure 9 shows the criterion.

[11]When the plant consists of one element, this equation can be obtained analytically.

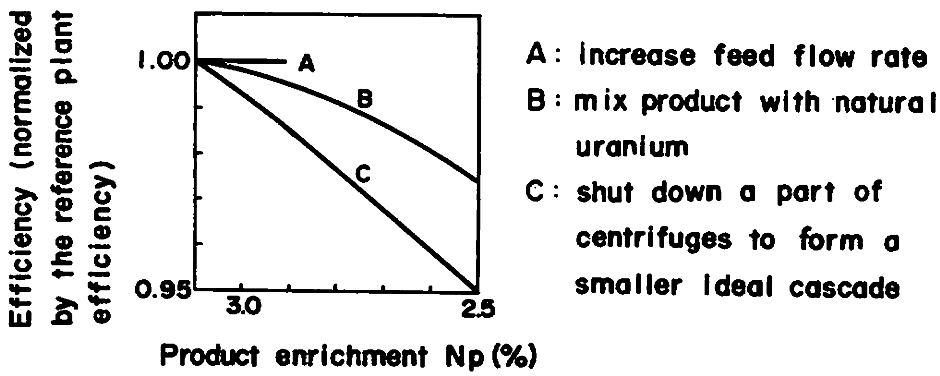

Fig. 8. Comparison among operational methods for varying product enrichment.

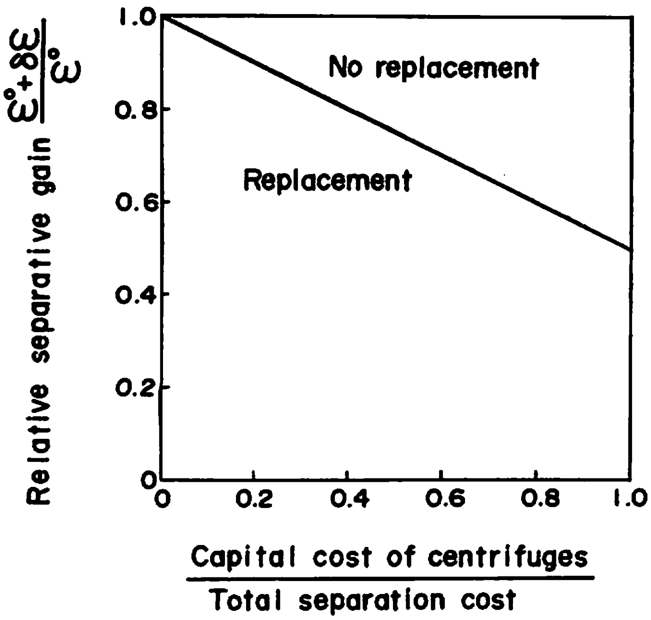

Fig. 9. Replacement criterion of a deteriorated centrifuge.

V. CONCLUSIONS

The characteristics of the ideal cascade are analyzed. The relations among the controlling variables [stage parameters (cut and separation gain) and feed flow rate] and the control variables (flow rate, enrichment and the separative work) are described by two differential equations. Then the perturbations are introduced and solved analytically for the first order approximation. Table II summarizes the results of the sensitivity expressions. Of course the validity of the conclusion is limited to the small perturbations owing to the linearization.

It was shown by the present study that the change in the feed flow and cut do not influence the separative work of the cascade, while the change in the separative work of a constituent centrifuge contributes directly to the cascade system without the mixing loss. This is one of the simplifying features of the ideal cascade. We have illustrated the use of these features in the considerations of a few operational problems.

Considering the stochastic nature of the plant performance,[10] as caused by the failure of centrifuges, tolerance of cut, and control error, it is not sufficient to treat the parameters deterministically.

The authors hope that the expressions for the sensitivity analysis presented in this paper will help better understanding of the ideal cascade for the design and operation of the uranium enriching centrifuge plants.

TABLE II

Equation Numbers Relating the Controlling and Controlled Variables

Controlling	Controlled						
	δF	δF_P	δF_w	δN	δN_P	δN_w	δU
δF_f	34	34	34	36	38	38	39
$\delta\Theta$	41	42	43	--	43	43	0
$\delta\epsilon$	0	0	0	30[a]	32	32	33

[a] δG instead of δN.

APPENDIX A

Derivation of Equation (13)

The conservation of $^{235}UF_6$ demands that

$$N_i F_i = N_i' F_i' + N_i'' F_i'' \quad . \tag{A.1}$$

Approximation (8) enables N' and N'' to be written as

$$N' = (1 + \xi)N$$
$$N'' = (1 - \eta)N$$
$$\xi, \eta \ll 1 \quad .$$

Using Eq. (5), primes are eliminated, and a relation between ξ and η follows:

$$1 = \theta(1 + \xi) + (1 - \theta)(1 - \eta) \quad , \text{ or } \quad \theta\xi = \bar{\theta}\eta \quad . \tag{A.2}$$

Another relation between ξ and η comes from Eqs. (6), (7), and (9),

$$(1 + \xi) = (1 - \eta)(1 + 2\epsilon) \quad , \text{ or } \quad \xi + \eta = 2\epsilon \quad . \tag{A.3}$$

Solution of ξ and η from Eqs. (A.2) and (A.3) proves directly Eq. (13).

APPENDIX B

Solution of Equations (15a) and (15b)

Equation (15a):

$$\frac{d^2}{dx^2} F + \epsilon^0 \frac{d}{dx} F = 0 \qquad 0 \le x < x_0 \ , \quad x_0 < x \le l \tag{B.1}$$

is integrated to give

$$F_- = A_1 + C_1 \exp(-\epsilon^0 x) \quad 0 \le x < x_0$$
$$F_+ = A_2 + C_2 \exp(-\epsilon^0 x) \quad x_0 < x \le l \quad . \tag{B.2}$$

Here four integration constants are determined by the boundary conditions, a continuity condition, and a jump condition.

$$\left.\begin{array}{ll} F_- = 0 & x = 0 \\ F_+ = 0 & x = l \\ F_- = F_+ & x = x_0 \\ \frac{d}{dx}F_+ - \frac{d}{dx}F_- = -2F_f & x = x_0 \quad . \end{array}\right\} \tag{B.3}$$

The last condition follows from integration of Eq. (15a) in the small range $(x_0 - \delta, x_0 + \delta)$,

$$-\frac{1}{2}\frac{d}{dx}F\Big]_{x_0-\delta}^{x_0+\delta} - \frac{1}{2}\epsilon^0 F\Big]_{x_0-\delta}^{x_0+\delta} = F_f \quad , \tag{B.4}$$

where the second term vanishes as δ tends to zero. By a straightforward calculation the coefficients are solved and the solution (16a) is obtained.

Equation (15b) is identical to Eq. (15a) if the following substitutions are made:

$$\begin{array}{l} \epsilon^0 \longrightarrow -\epsilon^0 \\ F_f \longrightarrow N_f F_f \quad . \end{array} \tag{B.5}$$

Hence the solution follows from Eq. (16a) by the above substitutions.

APPENDIX C

Derivation of Equations (20) and (21)

In this Appendix symbols with the suffix j denote the various quantities for a centrifuge within a stage; symbols having no suffix denote the various quantities for a stage. The relations (5) and (13) hold for each centrifuge:

$$F_j' = \theta_j F_j \tag{C.1}$$

$$G_j' = N\{1 + 2(1 - \theta)\epsilon_j\}\theta_j F_j \quad . \tag{C.2}$$

The parameters and the inlet flow fluctuate around the nominal values

$$\left.\begin{array}{l} \theta_j = \theta^0 + \delta\theta_j \\ \epsilon_j = \epsilon^0 + \delta\epsilon_j \\ F_j = F^0 + \delta F_j \quad . \end{array}\right\} \tag{C.3}$$

Inserting (C.3) into (C.1) and (C.2), and neglecting higher order terms

$$\begin{aligned} F_j' &= \theta^0 F^0 + F^0\delta\theta_j + \theta^0\delta F_j \\ G_j' &= NF_j' + 2N\{\theta^0\overline{\theta^0}\epsilon^0F^0 + (1 - 2\theta^0)\,\epsilon^0F^0\delta\theta_j \\ &\quad + \theta^0\overline{\theta^0}F^0\delta\epsilon_j + \theta^0\overline{\theta^0}\epsilon^0\delta F_j\} \quad , \end{aligned} \tag{C.4}$$

the stage flow is obtained by summing the flow of constituent centrifuges

$$\begin{aligned} F' &= \theta^0 F + F\langle\delta\theta_j\rangle + \theta^0\delta F \\ G' &= NF' + 2N\{\theta^0\overline{\theta^0}\epsilon^0F^0 + (1 - 2\theta^0)\epsilon^0F\langle\delta\theta_j\rangle \\ &\quad + \theta^0\overline{\theta^0}F\langle\delta\epsilon_j\rangle + \theta^0\overline{\theta^0}\epsilon^0\delta F\} \quad . \end{aligned} \tag{C.5}$$

On the other hand, stage flow is described by stage cut and stage separative gain as

$$F' = (F + \delta F)(\theta + \delta\theta)$$

$$G' = N\{1 + 2(1 - \theta - \delta\theta)(\epsilon + \delta\epsilon)\}(\theta + \delta\theta)(F + \delta F) \quad . \tag{C.6}$$

Equations (C.6) and (C.5) agree if $\delta\theta$ and $\delta\epsilon$ are defined by

$$\delta\theta = \langle\delta\theta_j\rangle \quad , \quad \delta\epsilon = \langle\delta\epsilon_j\rangle \quad . \tag{C.7}$$

These are the equations to be derived.

APPENDIX D

Derivation of Equation (38)

In Eq. (37), the following equalities obtained from Eq. (17) are inserted:

$$\epsilon^0 x_0 = \log\frac{N_f}{N_w} \quad , \quad \epsilon^0 l = \log\frac{N_P}{N_w} \quad , \quad \text{etc.}$$

Then F_w, F_P, F_f are introduced from Eq. (18). Thus

$$\begin{aligned} \frac{\delta N_w}{N_w} &= \Delta\left(\epsilon^0 x_0 + \frac{\epsilon^0 l}{\exp(\epsilon^0 l) - 1} - \frac{\epsilon^0(l - x_0)}{\exp[\epsilon^0(l - x_0)] - 1}\right) \\ &= \frac{\Delta N_P}{N_P - N_f}\frac{1}{F_f}(F_f \log N_f - F_P \log N_P - F_w \log N_w) \quad . \end{aligned}$$

Noting that $-\log N$ is the value function, the quantity in ( ) is just the separative work of the plant, which is related to the total number of centrifuges by Eq. (23):

$$(\ ) = \tfrac{1}{2}\epsilon^2 f J_T = U_T \quad . \tag{D.1}$$

This is the separative work of the cascade. Thus the first equation of Eq. (38) readily follows, and so does the second.

APPENDIX E

On the Postulate of Separative Work Stationarity

In the text we have postulated that the separative work is stationary in an ideal cascade, and proceeded to find the first order change in the product and waste isotopic assays resulting from a small cut change $\delta\theta_i$ at stage i. However, promoted by the referee's suggestion, we could prove the stationarity as follows.

Equation (28) for the separative work

$$\delta U = -F_P\frac{\delta N_P}{N_P} - F_w\frac{\delta N_w}{N_w} - \log\frac{N_P}{N_f}\delta F_P - \log\frac{N_w}{N_f}\delta F_w \tag{E.1}$$

is expressed in terms of δG_P and δF_P as

$$\delta U = \left(\frac{1}{N_w} - \frac{1}{N_P}\right) \delta G_P - \log \frac{N_P}{N_w} \delta F_P \quad , \tag{E.2}$$

where relations

$$\left.\begin{aligned} N = \frac{G}{F} \ , \quad \frac{\delta N}{N} = \frac{\delta G}{G} - \frac{\delta F}{F} \\ \delta F_P + \delta F_w = 0 \ , \quad \delta G_P + \delta G_w = 0 \end{aligned}\right\} \tag{E.3}$$

are used. The second term in Eq. (E.2) is readily obtained from Eq. (42). The first term, δG_P, consists of two terms representing direct and indirect effects. The indirect term represents the contribution of separative gain change, $\delta\epsilon_j$, induced by stage flow change δF_j throughout the cascade ($i = 0 \sim l$) caused by a stage cut change $\delta\theta_i$.

The direct term, $\delta G_P^{(d)}$, is obtained from Eq. (29) in a similar way as Eq. (30) was derived.

$$\begin{aligned} \delta G_{l-1} &= \frac{4\epsilon^0 G_i \exp(-\epsilon^0 x_i)}{1 - \exp(-\epsilon^0 l)} \delta\theta_i \\ \delta G_P^{(d)} &= \theta\delta G_{l-1} = \frac{2\epsilon^0 G_i \exp(-\epsilon^0 x_i)}{1 - \exp(-\epsilon^0 l)} \delta\theta_i \quad , \end{aligned} \tag{E.4}$$

where approximations $\theta = \bar{\theta} = \frac{1}{2}$ have been used.

As for the indirect term, the change in separative gain of various stages k is obtained from Eqs. (24) and (41) as

$$\begin{aligned} \delta\epsilon_k &= C_{ik}[1 - \exp(-\epsilon^0 x_k)]\,\delta\theta_i \qquad x_k < x_i \\ \delta\epsilon_k &= C_{ik}\{1 - \exp[\epsilon^0(l - x_k)]\}\,\delta\theta_i \qquad x_k > x_i \\ C_{ik} &= \frac{2\epsilon^0 \exp(\epsilon^0 x_i) F_i}{F_k[\exp(\epsilon^0 l) - 1]} \quad . \end{aligned} \tag{E.5}$$

The contribution of $\delta\epsilon_k$ to δG_P is expressed in Eq. (30):

$$\delta G_P = \frac{\epsilon^0 \exp(-\epsilon^0 x_k)\, G_k\, \delta\epsilon_k}{1 - \exp(-\epsilon^0 l)} \quad . \tag{E.6}$$

From Eqs. (E.5) and (E.6) the indirect term is composed by a summation,

$$\begin{aligned} \delta G_P^{(i)} &= \frac{2(\epsilon^0)^2 \exp(\epsilon^0 x_i)\, F_i\, \delta\theta_i}{[1 - \exp(-\epsilon^0 l)]\,[\exp(\epsilon^0 l) - 1]} \\ &\times \left(\sum_{k=0}^{i} \frac{\exp(-\epsilon^0 x_k)\, G_k[1 - \exp(-\epsilon^0 x_k)]}{F_k} \right. \\ &\left. + \sum_{k=i+1}^{l} \frac{\exp(-\epsilon^0 x_k)\, G_k\{1 - \exp[\epsilon^0(l - x_k)]\}}{F_k}\right) \quad . \end{aligned} \tag{E.7}$$

Using the relations $G_k = N_k F_k$, $N_k = N_l \exp[\epsilon^0(x_k - x_0)]$, and replacing the summation by integration, Eq. (E.7) is further simplified as

$$\begin{aligned} \delta G_P^{(i)} &= \frac{2(\epsilon^0)^2\, G_i\, \delta\theta_i}{[1 - \exp(-\epsilon^0 l)]\,[\exp(\epsilon^0 l) - 1]} \\ &\times \left(\sum_{k=0}^{i} [1 - \exp(-\epsilon^0 x_k)] \right. \\ &\left. + \sum_{k=i+1}^{l} \{1 - \exp[\epsilon^0(l - x_k)]\}\right) \\ &= \left\{\frac{2\epsilon^0 G_i\, \epsilon^0 l}{[1 - \exp(-\epsilon^0 l)]\,[\exp(\epsilon^0 l) - 1]} \right. \\ &\left. + \frac{2\epsilon^0 G_i \exp(-\epsilon^0 x_i)}{1 - \exp(-\epsilon^0 l)}\right\} \delta\theta_i \quad . \end{aligned} \tag{E.8}$$

Adding Eqs. (E.4) and (E.8),

$$\begin{aligned} \delta G_P &= \frac{2\epsilon^0 G_i\, \epsilon^0 l}{[1 - \exp(-\epsilon^0 l)]\,[\exp(\epsilon^0 l) - 1]} \delta\theta_i \\ &= \frac{2\epsilon^0 G_i N_P N_w \log(N_P/N_w)}{(N_P - N_w)^2} \delta\theta_i \quad , \end{aligned} \tag{E.9}$$

where $\epsilon^0 l = \log \frac{N_P}{N_w}$ has been used.

Using Eqs. (E.9) and (42) for δG_P and δF_P, Eq. (E.2) is finally reduced as

$$\begin{aligned} \delta U &= \left(\frac{1}{N_w} - \frac{1}{N_P}\right) \frac{2\epsilon^0 G_i N_P N_w \log(N_P/N_w)}{(N_P - N_w)^2} \delta\theta_i \\ &- \log\left(\frac{N_P}{N_w}\right) \frac{2\epsilon^0 G_i}{N_P - N_w} \delta\theta_i = 0 \quad , \end{aligned} \tag{E.10}$$

which is the result to be proved.

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