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Technical Notes

Optimization of Control Rod Programming and Loading Pattern in Multiregion Nuclear Reactor by the Method of Approximation Programming

Hiroshi Motoda

Atomic Energy Research Laboratory, Hitachi, Ltd.
Ozenji, Kawasaki, Kanagawa-Pref., Japan

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ABSTRACT

A nonlinear programming technique was applied to a one-dimensional multiregion slab reactor to optimize control rod programming and fuel loading pattern simultaneously.

Original equations and constraints which were continuous in space and time were discretized and further linearized to use linear programming repeatedly.

Numerical results were confirmed by the previously developed two-region burnup space theory. Furthermore, the more quantitative evaluation of burnup optimization and the determination of more realistic control rod programming became possible by the increased degree of control freedom.

I. INTRODUCTION

If it is possible to attain higher burnups of fuels discharged from nuclear power reactors or to lower required enrichment of fresh fuels by an appropriate control of power shape by means of a control rod programming and a fuel loading pattern, considerable economy can be achieved in a nuclear power station which will result in the reduction of power generating cost.

Terney and Fenech[1] solved the problem of determining an optimal sequence of control rod motions in a representative pressurized water reactor using dynamic programming and direct flux-synthesis. Wade and Terney[2] posed the design and operation of a nuclear reactor as optimal control problems by use of a generalized set of design objectives and a generalized control, and worked out the iterative algorithm by the gradient method and linear programming approach to solve a set of equations obtained as the necessary conditions for optimality by use of the Pontryagin's Maximum Principle. Mélice[3] developed a method to find the enrichment of fresh fuel and the location patterns of various assemblies of the SENA reactors by synthesizing an optimal distribution of nuclear property that maximizes the reactivity. There have also been many other contributions[4-10] in this field.

The author developed a theory of optimal control-rod programming for a two-region reactor model and several important results have been obtained.[11,12] A significant characteristic of burnup optimization problem is that an optimal terminal state (OTS) can be uniquely determined depending on an initial state and a control freedom. Suzuki and Kiyose[13] examined this nature more generally using a topological mapping theory. Therefore, the problem is focused on investigating the nature of the OTS and synthesizing an optimal control rod programming during one refueling interval.

The optimal terminal state satisfies the minimum material buckling condition within the constraint of power peaking factor, when the burnup dependence of material buckling is nearly linear in the two-region model. The solution of this problem shows the discontinuity in the optimal distribution of nuclear property of fuel, and the final state cannot be attained from any continuous, initial fuel distribution. However, this nature is shaded off in the two-region model, because the discrepancy of material buckling at the interface of two regions can be thought of as the effect of an averaging operation within a region. It is felt necessary to clarify this point and find a more realistic control rod programming by a more realistic model.

It is thought almost impossible to solve an optimal solution analytically for such a complicated system, by using the optimal control theory such as Pontryagin's Maximum Principle as is done in Ref. 11. We thought it may be useful to adopt a method of mathematical programming after several numerical experiments. As the first step, the analyses of a one-group, one-dimensional, multiregion slab reactor is attempted. No feedback reactivity and constraints of other than power peaking factor, fuel burnup, control rod density, and nuclear property of fuel are considered for simplicity.

II. STATEMENT OF THE PROBLEM

A one-dimensional, one-group neutron diffusion equation is written as

$$\left(\frac{2M}{H}\right)^2 \frac{\partial^2 \phi(x,t)}{\partial x^2} + [k_0(x) - 1 - \alpha(x)e(x,t) - u(x,t)]\phi(x,t) = 0 \quad , \tag{1}$$

where

H = width of a core
M = migration length

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[1] W. B. TERNEY and H. FENECH, *Nucl. Sci. Eng.*, **39**, 109 (1970).
[2] D. C. WADE and W. B. TERNEY, *Nucl. Sci. Eng.*, **45**, 199 (1971).
[3] M. MÉLICE, *Nucl. Sci. Eng.*, **37**, 451 (1969).
[4] I. WALL and H. FENECH, *Nucl. Sci. Eng.*, **22**, 285 (1965).
[5] R. L. STOVER and A. SESONSKE, *J. Nucl. Energy*, **23**, 673 (1969).
[6] J. R. FAGAN and A. SESONSKE, *J. Nucl. Energy*, **23**, 683 (1969).
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[10] T. KAWAI and T. KIGUCHI, *Nucl. Sci. Eng.*, **43**, 342 (1971).
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[13] A. SUZUKI and R. KIYOSE, *Nucl. Sci. Eng.*, **44**, 121 (1971).

$\phi(x,t)$ = neutron flux at location x and time t

$e(x,t)$ = burnup at location x and time t

$u(x,t)$ = control rod density at location x and time t

$k_0(x)$ = distribution of the infinite multiplication factor of the initial state

$\alpha(x)$ = depletion coefficient of reactivity by the fuel burnup process.

It is assumed that the decrease of the infinite multiplication factor with fuel burnup is linear and that the migration area M^2 is constant over the entire space and time. The boundary condition to Eq. (1) is

$$\frac{\partial\phi(0,t)}{\partial x} = 0 \quad , \qquad \frac{\partial\phi(1,t)}{\partial x} + \frac{1}{l}\,\phi(1,t) = 0 \quad , \tag{2}$$

where l is an extrapolated length of the core.

The burnup $e(x,t)$ is proportional to the accumulated reactor power. Thus, assuming the equivalence of neutron flux and power,

$$\frac{\partial e(x,t)}{\partial t} = \phi(x,t) \quad . \tag{3}$$

The boundary condition to Eq. (3) is

$$e(x,0) = 0 \quad . \tag{4}$$

As operational constraints, the total reactor power must be constant

$$\int_0^1 \phi(x,t)\,dx = 1.0 \quad , \tag{5}$$

and the power, the fuel burnup and the control rod density should be less than the allowable values

$$0 \le \phi(x,t) \le f \tag{6}$$

$$0 \le e(x,t) \le E \tag{7}$$

$$0 \le u(x,t) \le U \quad , \tag{8}$$

and finally the initial nuclear property of fuel must be within some given range,

$$k_{\min} \le k_0(x) \le k_{\max} \quad . \tag{9}$$

The average fuel burnup $\bar{e}$ at the final time t_f is

$$\bar{e} = \int_0^{t_f}\int_0^1 \phi(x,t)\,dx\,dt = t_f \quad , \tag{10}$$

and is proportional to the operational period t_f. If the unit of time t is taken as a year, $\alpha(x)$ means the reactivity depletion during one year.

The optimization problem is defined here as follows: to find the space-time variation of $u(x,t)$ that minimizes the amount of fuel loading while satisfying Eqs. (1) to (9) for the given t_f. Therefore, the performance index to this problem should be

$$J = \int_0^1 k_0(x)\,dx \quad . \tag{11}$$

This problem is equivalent to maximizing the average burnup of a given initial loading pattern if the relative distribution of $k_0(x)$ is fixed and $\alpha(x)$ is independent of x and positive as is evident by Eq. (12)

$$\int_0^1 \delta k_{t_f}\,dx = \int_0^1 \delta k_0\,dx - \alpha\bar{e} \quad , \tag{12}$$

where δk_0 and δk_{t_f} are reactivities of fuel at time 0 and t_f respectively. However, this formulation enables us to find an optimal loading pattern, i.e., an optimal distribution of nuclear property of fuel, as well as an optimal control rod programming. We call the former problem as problem one and the latter as problem two.

The assumptions of the equivalence of neutron flux and power and the linear depletion of the reactivity with fuel burnup are easily removed in the method described in the next section.

III. APPLICATION OF THE METHOD OF APPROXIMATION PROGRAMMING

This is one of the methods of solving a nonlinear programming problem. The original nonlinear problem is reduced to a linear problem by expanding variables around a feasible solution, neglecting higher order terms. An optimal solution can be obtained by solving the linear programming (L.P.) repeatedly, starting from some promising initial solution, until the convergence becomes satisfied. This method was first developed by Griffith and Stewart.[14] Many have applied L.P. to nuclear reactor problems.[15-20] Frankowski[21] applied the method of approximation programming (MAP) to the optimization of the structure of a developing system of nuclear power stations. Mélice, Hunin, and Vielvoye[22] used this method to find an optimal configuration of fuels in pressurized water reactor cores. Wade and Terney also used this method in Ref. 2 to modify a control to make the Hamiltonian smaller.

Let one feasible solution be k_{0n}^0, $\phi_{n,m}^0$, $e_{n,m}^0$, and $u_{n,m}^0$ and the small change of each of these variables be k_{0n}, $\phi_{n,m}$, $e_{n,m}$, and $u_{n,m}$. Here the spatial coordinate x and the operational period t_f are divided into N - 1 and M - 1 small intervals respectively and only values at node (n,m) ($n = 1, \ldots N$, $m = 1, \ldots M$) are considered. Furthermore, let the number of control rod groups be Nu, the number of mesh points assigned to the i'th group be $Nd(i)$ and the j'th mesh number of the i'th group be $Kn(i,j)$. It is not necessary that $Kn(i,j)$ is continuous in j.

Then Eqs. (1) to (9), and (11) are discretized by central difference and forward difference approximations and further linearized to obtain a standard linear programming of the following form.

$$\left.\begin{aligned} &\sum_{i=1}^{i_0} a_{ji}\,x_i \gtreqless b_j && j = 1, \ldots j_0 \\ &x_i^l \le x_i \le x_i^u && i = 1, \ldots i_0 \\ &J = \sum_{i=1}^{i_0} c_i\,x_i \to \min && \end{aligned}\right\} \quad . \tag{13}$$

The correspondence of x_i to each variable is shown below.

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[15] D. TABAK, "Optimization of Nuclear Reactor Fuel Recycle via Linear and Quadratic Programming," *IEEE Trans. on Nuclear Science*, **NS-15**, 60 (1968).

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[19] Y. SHINOHARA, S. YASUKAWA, and J. SHIMAZAKI, *J. Nucl. Sci. Technol.*, **7**, 615 (1970).

[20] W. FRANKOWSKI, *Soviet Atomic Energy*, **27**, 396 (1969).

[21] W. FRANKOWSKI, *Soviet Atomic Energy*, **29**, 1071 (1970).

[22] M. MÉLICE, C. HUNIN, and A. VIELVOYE, "In-core Fuel Management in PWR Plant—A Practical Approach," *Proc. Intern. Conf. Peaceful Uses At. Energy, Geneva*, **2**, 611 (1971).

$$\left.\begin{aligned}
x_n &= k_{0n} & n &= 1,\dots L \\
x_{N(m-1)+n+L} &= \phi_{n,m} & n &= 1,\dots N \\
& & m &= 1,\dots M \\
x_{(N-1)(m-1)+n+NM+L} &= e_{n,m} & n &= 1,\dots N-1 \\
& & m &= 1,\dots M \\
x_{Nu(m-1)+i+2NM+L-M} &= u_{Kn(i,1),m} & i &= 1,\dots Nu \\
& & m &= 1,\dots M
\end{aligned}\right\} , \quad (14)$$

where

$$L = \begin{cases} 1 & \text{for problem one} \\ N-1 & \text{for problem two} \end{cases} .$$

Explicit expressions of a_{ji}, b_j, and c_i are given in the Appendix. The equations are numbered as they appear in Sec. I, with m assuming its complete range of values for fixed n, then n increasing. After n has assumed its maximum value, the expressions apply to the next equation.

It is necessary to prepare the upper and the lower bounds for each variable other than those given by the problem to assure the accuracy of linearization. Let these bounds be δk, $\delta\phi$, δe, and δu. Then the constraints for each variable are given as the common regions. Explicit expressions of these constraints are also given in the Appendix.

The numbers of variables i_0 and constraints $j_0 + i_0$ are $2NM + M(Nu - 1) + L$ and $M(4N + Nu) - M + L$ respectively. For example, if $N = 6$, $Nu = 5$, $M = 11$, and $L = 5$, these become 181 and 313.

All of Eqs. (A.2) should be zeroes if each variable satisfies the difference equations of Eqs. (1) to (5). However, these are not set at zeroes to avoid the accumulation of errors by the repeated linearizations. Thus, it is not necessary that any intermediate solution rigorously satisfies the relation that the right hand sides of Eqs. (A.2) be zeroes.

The convergency of this problem strongly depends on the nonlinearity of the diffusion equation and the initial guess. Operation by Haling's principle[23] and/or by the uniform control can be used as the initial solution.

IV. RESULTS AND DISCUSSIONS

This calculational method was applied to a typical boiling water reactor of 500 MW(e). The core width was determined so that the radial neutron leakage became nearly equal for both the slab and the cylindrical geometries. The main data used were as follows:

$$M^2 = 80\ \text{cm}^2 , \quad H = 200\ \text{cm} , \quad \alpha = 0.1$$
$$l = 0.0 , \quad t_f = 1.0\ \text{yr} , \quad U = 0.5$$
$$E = 2.0 , \quad k_{\min} = 0.9 , \quad k_{\max} = 1.2$$

The values of $k_{\min}$ and $k_{\max}$ are rather arbitrary. They were chosen to represent the values of fresh and burnt fuel.

A. *Conventional Control Rod Programming*

The control rod programmings by Haling's principle and the uniform control were calculated. These were used as the initial guesses and also as the standard for comparison with optimal solutions. These control rod programmings are obtained by solving Eqs. (15) and (16) respectively.

$$\left.\begin{aligned}
&\left(\frac{2M}{H}\right)^2 \frac{d^2\phi(x)}{dx^2} + [k_0(x) - 1 - \alpha(x) t_f\, \phi(x)]\phi(x) = 0 \\
&u(x,t) = \alpha(x)\phi(x)(t_f - t)
\end{aligned}\right\} \quad (15)$$

$$\left(\frac{2M}{H}\right)^2 \frac{\partial^2\phi(x,t)}{\partial x^2} + [k_0(x) - 1 - \alpha(x)\int_0^t \phi(x,t)\,dt - u(t)]\phi(x,t) = 0 \quad . \quad (16)$$

The results are shown in Fig. 1. The first two rows show the time variation of the power distribution $\phi(x,t)$ of the operation by Haling's principle and the uniform control respectively, and the last two show that of the corresponding distribution of the control rod density $u(x,t)$. The reactor core is divided into five meshes ($N_1 = 5$) and the control rod density is allowed to vary at each mesh ($Nu = 5$). The operational period t_f is divided into 11 meshes ($M = 11$) and only the results at $t = 0.0$, 0.5, and 1.0 yr are shown. The shaded area indicates the net reactivity $\delta k - u$. It is assumed in these examples that the fuel is loaded uniformly in the core and has flat nuclear property at the beginning of operation.

The power distribution does not change throughout the operational period in Haling's principle and the control rod density is proportional to the power density. On the other hand, the power distribution changes considerably in the uniform control. It becomes flatter as the fuel burns. The required δk_0 is a little larger than that of

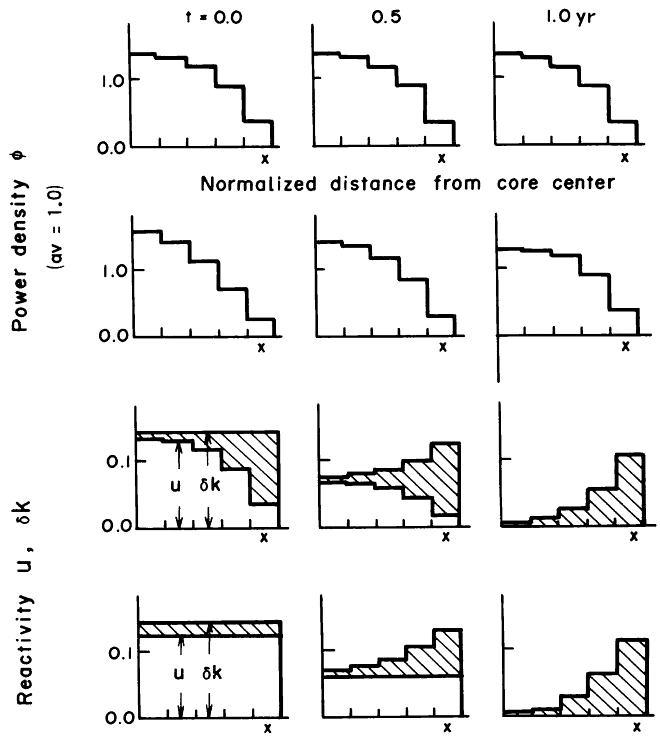

Fig. 1. Conventional control rod programming ($N_1 = 5$, $Nu = 5$, $M = 11$).

The first and the third rows refer to Haling's principle, and the second and the fourth rows refer to uniform control. Shaded area indicates net reactivity $\delta k - u$.

[23] R. K. HALING, "Operating Strategy for Maintaining an Optimal Power Distribution Throughout Life," *Proc. ANS Topical Meeting on Nuclear Performance of Power Reactor Cores*, TID-7672, U.S. Atomic Energy Commission (1964).

Haling's principle. This is because the higher power density in the central region promotes the fuel burnup in this region and the reactor has to be kept critical with the fuel of worse nuclear property in the central region. In both cases, the control rods are fully withdrawn at the end of operation. The difference of δk_0 is nearly equal to 2% of fuel burnup.

B. Optimal Control Rod Programming for Uniform Loading (Problem One)

The problem here defined is to find the OTS attainable from an initial state.

Effect of the maximum allowable power peaking factor f on the optimal solutions are investigated for the uniform loading. The relation between f and δk_0 is shown in Fig. 2. The ○ marks refer to the optimal solutions and △ and □ marks refer to Haling's principle and the uniform control respectively. These are calculated with $N_1 = 5$, $Nu = 5$, $M = 11$.

It is evident that the results of Haling's principle and the uniform control lie above the optimal δk_0 - f curve. Especially the uniform control results in δk_0 which is considerably larger than that of the optimal solution for the same value of f. This is equivalent to the reduction of burnup of about 8%. On the other hand, Haling's solution gives the close result to the optimal solution. However, it is not optimal although the difference of δk_0 is very small and is equivalent to the burnup difference of about 1.4%. In addition, this principle does not give the smallest power peaking factor f for the given burnup $\bar{e}$ or the given initial nuclear property of fuel δk_0. If the reactor is designed to have a margin of power peaking over that of Haling's principle, the effect of optimization is worth noting. If f is taken to be 1.4 (which seems natural for the bare slab reactor), the difference of δk_0 or $\bar{e}$ amounts to as much as 0.0048 or 3.5%. From this figure it is understood that the optimal solution is classified into three regions according to the range of $f : f \lesssim 1.32$, $1.32 \lesssim f \lesssim 1.75$, $1.75 \lesssim f$.

Some examples of the time variation of the power and the control rod density distributions are shown below for these three regions. The results of $f = 1.3$ are shown in Fig. 3 as the example of the first region. Three calculations were performed, varying the mesh spacings to see their effects on the optimal solution. The first and the fourth are the results of $N_1 = 10$, $Nu = 10$, $M = 6$, the second and the fifth are of $N_1 = 10$, $Nu = 5$, $M = 6$, and the third and the sixth are of $N_1 = 5$, $Nu = 5$, $M = 11$.

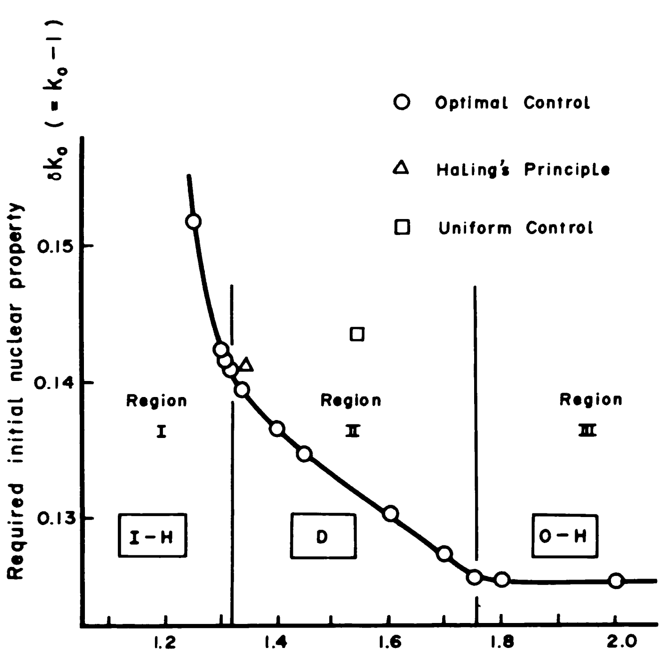

Fig. 2. Effect of maximum allowable power peaking factor f on required initial nuclear property of fuel δk_0 in case of the uniform loading, $N_1 = 5$, $Nu = 5$, $M = 11$, $U = 0.5$.

Optimal control rod programming is classified into three regions according to the range of f.

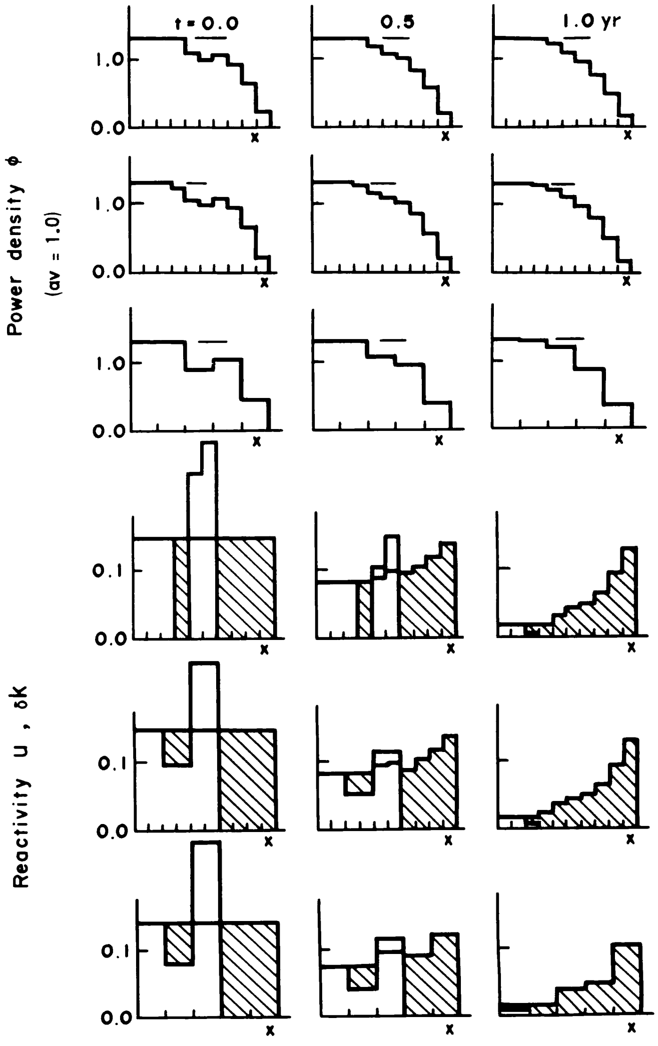

Fig. 3. Optimal control rod programming and power distribution, uniform loading, $f = 1.3$ (Region I).

Three results are shown. The first and the fourth rows are of $N_1 = 10$, $Nu = 10$, $M = 6$, the second and the fifth are of $N_1 = 10$, $Nu = 5$, $M = 6$, and the third and the sixth are of $N_1 = 5$, $Nu = 5$, $M = 11$. The optimal solution is unique and its policy is globally inner high and locally outer high.

The power distributions of the first two cases are very similar and well express the local variations. The third one is not enough to discuss the fine structure but still indicates the characteristics of the global variations. On the contrary, the control rod distributions seem considerably different between the first two cases, because its spatial distribution is much more irregular than the power distribution. However, the mesh-averaged rod densities are very close to each other. Thus, it is concluded that at least 5 spatial mesh and 6 time mesh points are required.

The ratios $\delta k:\delta\phi:\delta e:\delta u$ are fixed at 1:5:5:1 for all calculations in this study and $\delta\phi$ is chosen out of the values 0.01 to 0.05. The calculations of Fig. 3 are performed with $\delta\phi = 0.03$ and the number of the iterations of L.P. calculations are ~20. The initial guesses are the uniform control with the relative distribution of $u(x,t)$ chosen to satisfy Eqs. (6), (7), and (8) (modified uniform control).

The optimal policy of this region falls in inner high in the two-region model,[11,12] i.e., the power density of the inner region should be kept as high as possible at each instant. Indeed it is necessary to do so to minimize the residual control rod density at the end of operation. However, it is seen that the effort is paid to bring the nuclear property at the middle region as high as possible at the end of operation by making the power density at the outer region as high as possible. The control rod programming of the first case well explains this situation. The control rod density of the first three meshes is uniquely determined so as to make the power distribution flat, i.e., $k_\infty = 1.0$. At the next mesh the control rod is fully withdrawn to give the maximum material buckling, and then again inserted at the next two meshes to give the minimum material buckling. Finally at the last four meshes the control rod is fully withdrawn. This material buckling distribution makes the outer high power distribution at the outer region.

The control rod should be fully inserted to make the material buckling minimum and the optimal control rod distribution must be bang-pang type. This is not clear, however, in the above example. This may be because the optimal switching points lie inside the mesh intervals, perhaps, inside the fifth and sixth meshes. The control policy in this region is shown as inner high in Fig. 2. This is valid only in the global sense. We call this policy as globally inner high and locally outer high.

Figure 4 shows two examples of the second region. The maximum allowable power peaking factor f is set at the value of Haling's principle, 1.337, in the left example, and 1.6 in the right example. The optimal solution is degenerate and two different solutions are shown for each example. Haling's principle and the modified uniform control were used as the initial guesses. The iteration number was about 50 in these cases. The calculational condition is $N_1 = 5$, $Nu = 5$, $M = 11$, and $\delta\phi = 0.01$. This figure shows that the power density in the inner region must not necessarily be maximized throughout the operational period to withdraw all the control rods at the end of operation. Actually, inner high policy results in worse nuclear property in the inner region which requires more fuels to be loaded, and outer high policy results in better nuclear property in the inner region which requires some of the control rods still to remain inserted in the core to satisfy the constraint on power peaking factor. Thus the optimal solution must be between these two extreme policies and, therefore, degenerate. The power density of the central region is kept maximum and is equal to its limitation f at the end of operation and all of the control rods are fully withdrawn.

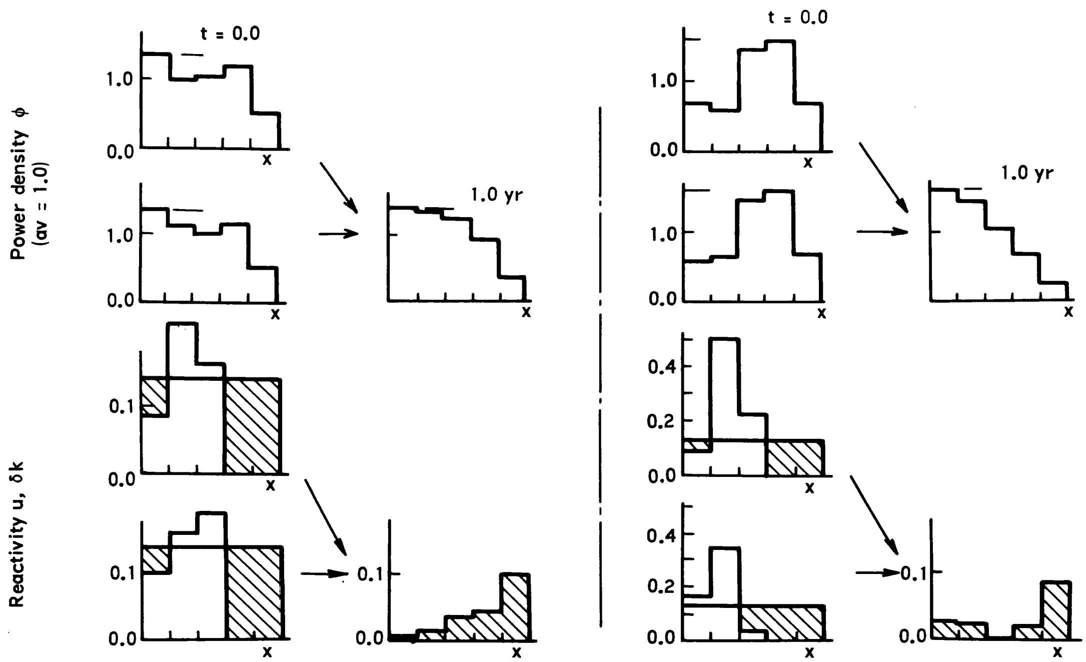

Fig. 4. Optimal control rod programming and power distribution, uniform loading, $f = 1.337$ (left), $f = 1.6$ (right), $N_1 = 5$, $Nu = 5$, $M = 11$ (Region II).
The optimal solution is degenerate and two solutions with different initial guesses are shown, both resulting in the same terminal states.

Haling's principle can be optimal in the two-region model where there is only one freedom for operation, but not in the multiregion model. It is optimal in the sense[23] that the power peaking factor is maintained at the minimum value for any given set of end conditions when the power shape does not change during the operational period.

The problem defined here is: to find the OTS and its resulting better nuclear property distribution at the end of operation than that of Haling's principle by the appropriate control rod programming without violating the constraint on f. This means in other words that better power distribution can be realized, giving the same average burnup as that of Haling's principle; however, Haling's principle should be evaluated by its high practicality.

An example of approach to convergence is shown in Fig. 5. The convergence is very slow for these degenerate cases, although the performance index rapidly falls to around the optimal value. The result of each iteration satisfies all of the constraints and thus the iteration can be terminated at any stage before the final convergence is met if the result is acceptable.

There is no unique policy of the control rod programming for this region, although it is near outer high and therefore, it is difficult to synthesize it as a function of the state of the reactor. Furthermore, it should be noted that this degenerate region covers the practical range of operational condition.

Figure 6 shows two examples of the third region. The first and the third rows refer to the uniform loading. This is solved for f = 1.8, but the power peaking factor never exceeds this value during the operational period because of the limitation of the control rod density U and thus, this is valid for $f \gtrsim 1.8$. The optimal solution is unique and its policy is theoretically outer high. The control rods are inserted from the core center and withdrawn from their outer surface as the time proceeds. The second and the fourth rows refer to the nonuniform loading. This is solved for Nu = 2, and falls under this region, too.

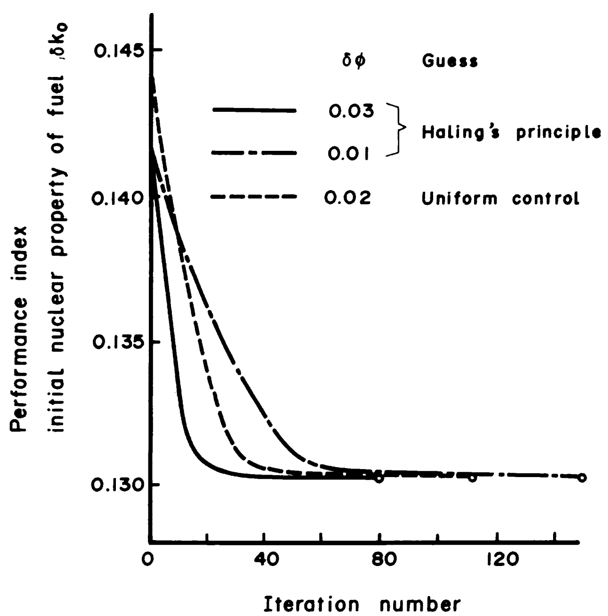

Fig. 5. Approach to convergence, uniform loading, f = 1.6, N_1 = 5, Nu = 5, M = 11.

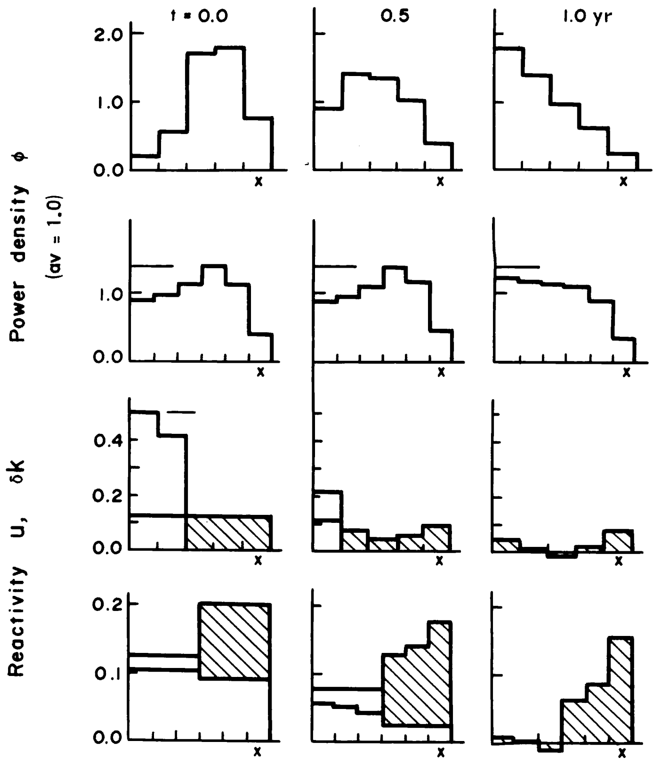

Fig. 6. Optimal control rod programming and power distribution (Region III).

The first and the third rows refer to uniform loading, $f \gtrsim 1.8$, N_1 = 5, Nu = 5, M = 11 and the second and the fourth rows refer to nonuniform loading, f = 1.4, N_1 = 6, Nu = 2, M = 11. The optimal solution is unique and its policy is outer high.

The computer running time for these calculations are about 2 min for the first L.P. calculation and thereafter, about 9 sec for each L.P. calculation by HITAC 5020F (equivalent to IBM 360/65). Therefore, the total running time is about 10 min for the iteration number of 50.

C. Optimal Control Rod Programming for Non-Uniform Loading (Problem One)

The optimal control rod programmings for the nonuniform, two-region outer high loading were calculated for f = 1.4. The discrepancy of the nuclear property $\Delta(\delta k_0)$ was varied from 0.1 to 0.2. The minimum required value of δk_0 averaged over the core increases as $\Delta(\delta k_0)$ becomes larger. The effect of the fuel loading pattern is very large and evidently it is disadvantageous to load better fuels in the outer region. Optimal control rod programmings are degenerate for $\Delta(\delta k_0) \lesssim 0.15$. The control rods are fully withdrawn at the end and the power peak moves outward as $\Delta(\delta k_0)$ becomes larger. Optimal control rod programming is unique and outer high for $\Delta(\delta k_0)$ = 0.2, and control rods still remain inserted in the middle region at the end of operation because of the high nuclear property at this region and the constraint on f.

D. Comparison with the Two-Region Model

To evaluate the effect of multiregion control quantitatively, the same problems were solved by the two-region control, i.e., Nu = 2, and the two-region model.[11] Two cases of the uniform loading and the nonuniform, two-region outer high loading with $\Delta(\delta k_0)$ of 0.1 were chosen for comparison, and f was set at 1.4 for the optimal solution. The results are shown in Table I. The relative difference of average δk_0 of the optimal solution from Haling's principle is shown for each case. The results of the two-region control and the two-region model are in good agreement and it is shown that the effect of the optimal control rod programming by multiregion control is about 3 times larger than the two-region control or the two-region model. Optimal solution for the uniform loading is degenerate for each of the three cases but optimal solution for the nonuniform loading is degenerate only for the multiregion control. When the power peak takes place in the outer region (nonuniform loading), the difference of δk_0 is smaller although the difference of power peaking factor is larger. This result is understandable from the gradient of critical curve in the two-region burnup space.[11]

TABLE I

Comparison with the Two-Region Model, f = 1.4, M = 11

Loading Pattern	N_1	N_u	Average δk_0		f_{Hal}	$\Delta\delta k_0$[a]
			Haling	Optimal	Haling	%
Uniform	6	6	0.1419	0.1364	1.345	3.9
	6	2		0.1400		1.3
	TRM[b]		0.1349	0.1328	1.335	1.6
Non-uniform	6	6	0.1526	0.1477	1.249	3.2
	6[c]	2		0.1512		0.9
	TRM		0.1442	0.1430	1.310	0.8

[a] $\Delta\delta k_0 = [(\delta k_0\text{Hal} - \delta k_0\text{Opt})/\delta k_0\text{Hal}] \times 100$.

[b] Two-region model.[11]

[c] This is shown in Fig. 6.

E. Optimal Control Rod Programming and Optimal Loading Pattern (Problem Two)

The optimal control rod programmings so far obtained are for the fixed loading pattern. How should the nuclear property be distributed in the core and how should the control rods be withdrawn to give the maximum burnup or to make the total amount of the fissile material minimum?

It is conjectured that this fuel distribution may be similar to the solution of the minimum fuel integral in the classical problem, because the optimal terminal state is one which gives the minimum fuel integral among these attainable from the initial state.

The solutions of the minimum fuel integral are shown in Fig. 7. The upper figure is a case where the variation of the macroscopic fission cross section Σ_f is assumed to be linear to k_∞ as $\Sigma_f = 0.165 + 0.835\ k_\infty$ and the lower figure is a case where it is set at 1.0 regardless of k_∞, which corresponds to the present treatment. Both of them are calculated by MAP with f = 1.4, l = 0.1, and N_1 = 40.

It is not necessary to bound k_∞ because the constraint

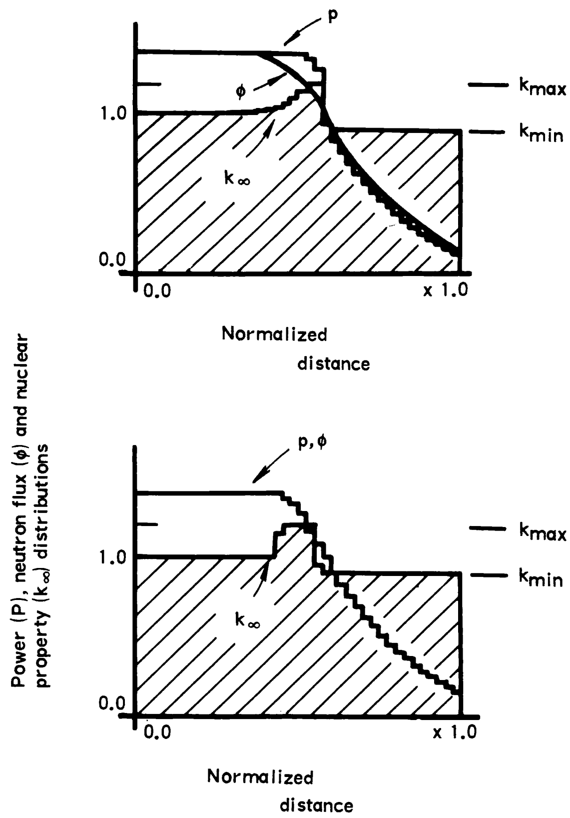

Fig. 7. Minimum fuel integral, f = 1.4, k_{min} = 0.9, k_{max} = 1.2, l = 0.1, N_1 = 40.

The upper figure is the case where the variation of Σ_f with k_∞ is taken into account as $\Sigma_f = 0.165 + 0.835\ k_\infty$ and the lower figure is the case where it is set at 1.0, which corresponds to the present treatment.

for power peaking is given; however, the same constraint was imposed as is given at the beginning of this section.

The optimal solution is very simple and three-region bang-pang type in the lower case. The nuclear property k_∞ is equal to 1.0 in the inner region, k_{max} in the middle region and k_{min} in the outer region. This solution is modified a little when the variation of Σ_f with k_∞ is taken into account as in the upper case. In this case, the k_∞ distribution in the inner region is not flat but is a smoothly increasing function of x such that $\Sigma_f\phi$ is a constant.

This solution is similar to that obtained by Zaritskaya and Rudik,[24] but the problem differs from theirs in that the constraint on the power peaking factor f is not the severest in our problem. More accurate treatment, i.e., multigroup treatment, will indicate the existence of the singular solution characterized by an analytical maximum of the Hamiltonian between regions two and three (four region loading). The k_∞ will decrease monotonically in this region.

The solutions of the optimal loading pattern and the optimal control rod programming are shown in Fig. 8. These are calculated for f = 1.4. The first and the third rows are the results of N_1 = 5, Nu = 5, M = 11, and the second and the fourth are of N_1 = 9, Nu = 9, M = 6.

[24] T. S. ZARITSKAYA and A. P. RUDIK, *Soviet Atomic Energy*, **22**, 5 (1967).

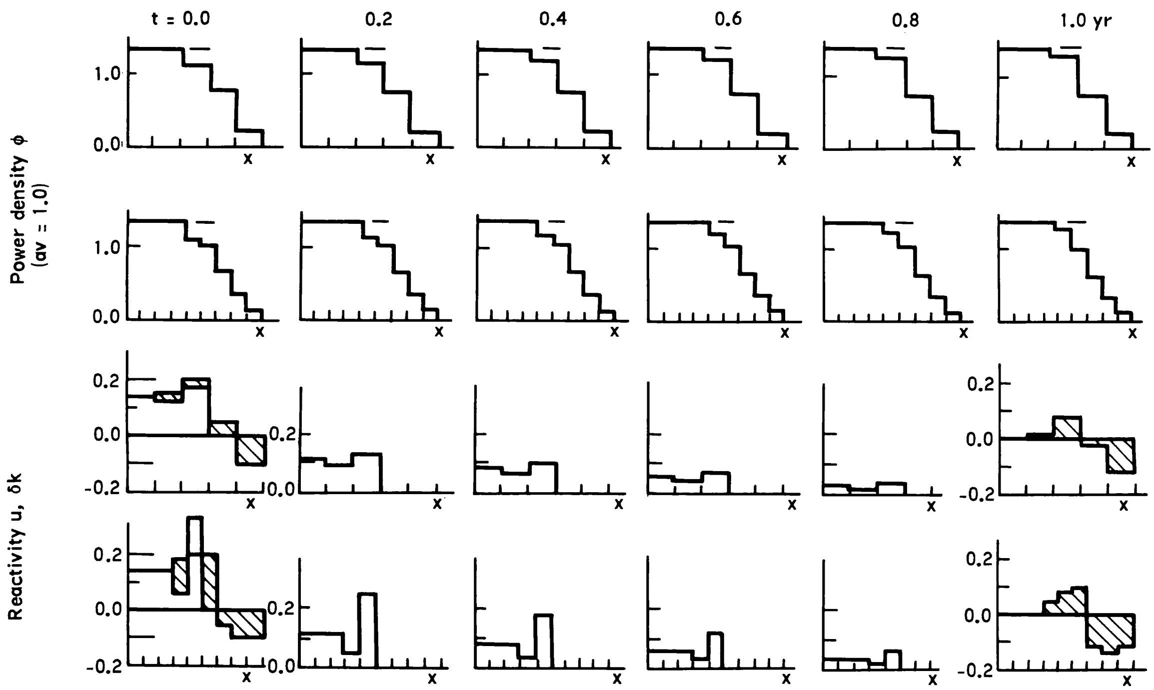

Fig. 8. Optimal loading pattern and optimal control rod programming, $f = 1.4$, $k_{min} = 0.9$, $k_{max} = 1.2$.
The first and the third rows are of $N_1 = 5$, $Nu = 5$, $M = 11$ and the second and the fourth are of $N_1 = 9$, $Nu = 9$, $M = 6$. The optimal loading pattern is a three region bang-pang type and the control rod programming is unique and similar to that of Fig. 3.

It is shown that the optimal loading pattern is also three-region bang-pang type. The power distribution and the control rod programming are unique and very similar to those in Fig. 3. The nuclear property of the inner region is uniquely determined by the maximum allowable power peaking factor f. The power density in this region is flat and maximum, and the control rod density is also flat giving the net nuclear property k_∞ of 1.0 during the whole operational period, and is reduced to zero at the end of operation. The nuclear property of the middle and the outer region is fixed at k_{max} and k_{min} respectively. The control rod density at these regions is such as to make the power distribution as outer high as possible. Thus, the control policy can be called globally inner high and locally outer high. The locations of the boundary of these regions must be determined by considering the attainability, i.e., the possibility that the reactor can be operated for the given period without violating the operational constraints. The volumes of these three regions are nearly equal. The optimal control rod programming seems to be unique as far as the optimal nuclear property required in the inner region lies between k_{max} and k_{min}. The distribution of δk_{t_f} in Fig. 8 is the OTS which depends on the limitations on the available nuclear property and operational history and thus, this cannot be calculated in advance without solving the burnup problem. More accurate treatment will result in the similar distribution of the nuclear property (four region loading) to that of the minimum fuel integral.

The OTS is uniquely determined for every initial fuel loading pattern and the operational constraints, but the control rod programming is not necessarily unique. There seems to be no practical method of finding the OTS and synthesizing the control rod programming as a function of the state vector, not of the time especially when the control policy is degenerate.

Haling's principle can be optimal for the optimal fuel loading pattern in the two-region model but not in the multiregion model. Haling's solution calculated by the time reversal analysis using the power and the nuclear property distributions at the end of operation in Fig. 8 results in k_∞ in the middle region larger than k_{max} and violates its limitation.

The effect of the optimal loading pattern and the optimal control rod programming was investigated and the results are shown in Table II. Haling's principle for the optimal loading pattern results in only 1% reduction of the operational period or 0.0013 increment of δk_0. The uniform loading, k_∞ of which is set at the average of the optimal loading pattern, results in as much as 45% reduction of

TABLE II

Effect of Optimal Loading Pattern and Optimal Control Rod Programming, $f = 1.4$, $k_{min} = 0.9$, $k_{max} = 1.2$, $N_1 = 5$, $Nu = 5$, $M = 11$

Loading Pattern	Rod Programming	Δt_f [a]	Δk_0 [b]
Optimal	Optimal	0.0	0.0
Optimal	Haling	-0.01	0.0013
Uniform[c]	Haling	-0.45	0.0503

[a]Difference of the operational period (year).

[b]Difference of the average initial nuclear property of fuel, t_f being fixed at 1.0 yr.

[c]Core averaged value conserved for the optimal solution.

the operational period or 0.0503 increment of δk_0. This difference is very large.

This loading effect was examined by an initial core of an actual representative boiling water reactor using FLARE code.[25] Haling's solutions are calculated for both the uniform and the nonuniform three-region loading patterns. The latter pattern is shown in Fig. 9. The average nuclear property of the three-region loading is chosen to be equal to that of the uniform loading. The results are shown in Table III. The three-region loading results in the burnup are about 20% larger than that of the uniform loading, the thermal characteristics being improved a little. The reactivity saving of 0.1% is obtained by the three-region loading.

Needless to say, an optimal fuel management should be discussed in the entire reactor life and it is a typical multistage decision process. There is no reason to justify that the same refueling schedule is repeated in every cycle. It seems reasonable and practical to assume that the region averaged nuclear properties be fixed at the optimal values such as is obtained in this study. Once this is distributed optimally, the optimal control rod programming is unique and the optimal allocation of each fuel assembly is rather straightforward. The methods developed by Mélice,[3] Naft et al.,[7] and Suzuki et al.[13] are useful.

V. CONCLUSION

The optimal control rod programming and the optimal loading pattern were determined for a one-group, one-dimensional, multiregion slab reactor by using the method of approximation programming. The original equations formalized as the optimization problem of a distributed parameter system were discretized and linearized, and the standard L.P. calculations were performed repeatedly. The numerical results were confirmed by the past two-region burnup space study. The increased freedom of control made it possible to cover more region than two, and many interesting results have been obtained. Some of the main results are summarized below.

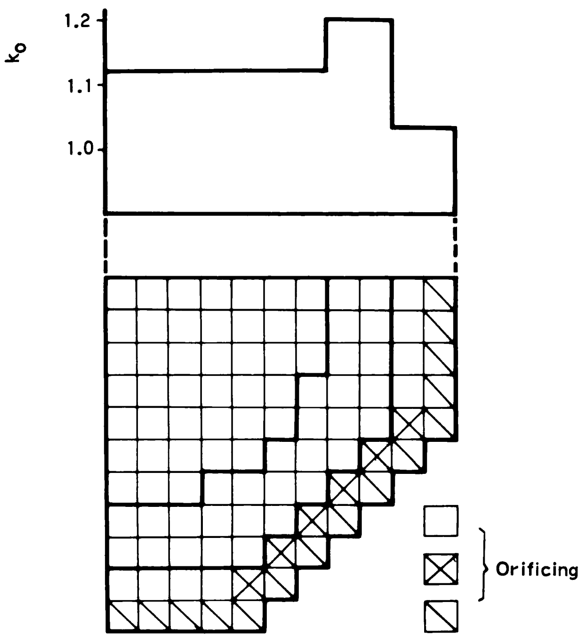

Fig. 9. Application of the three region loading principle to a three dimensional commercial BWR.

[25] D. L. DELP, D. L. FISHER, J. M. HARRIMAN, and M. J. STEDWELL, "FLARE . . . A Three-Dimensional Boiling Water Reactor Simulator," GEAP-4598, General Electric Co. (1964).

TABLE III

Comparison of the Fuel Loading Patterns by FLARE Code

	Uniform Loading	Three Region Loading
$\bar{e}$ (GWD/T)	5.58	6.79
$\Delta\bar{e}$ (%)		21.7
f	2.523	2.398
MLHR[a] (kW/ft)	13.211	12.557
MCHFR[b]	3.310	3.482
$\bar{k}_\infty$[c]	1.0448	1.0348

Control rod programming is determined by Haling's principle

[a] Maximum linear heat rate.
[b] Minimum critical heat flux ratio.
[c] Core averaged value of the target distribution.

1. The nature of the optimal control rod programming is classified in the following three types.

   a. The solution is unique when some of the control rods have to remain inserted at the end of the operation.

   b. The solution is unique when all of the control rods can be fully withdrawn and the power peaking factor is below the constraint at the end of operation.

   c. The solution is degenerate when the power peaking factor is equal to the constraint in case of type (b) in general. This can be unique only for the optimal loading pattern and for the special situations of the limit of types (a) and (b).

2. Although the optimal terminal state is uniquely determined for every initial fuel loading pattern and operational constraints, the control rod programming is not necessarily unique. No simple way of finding this target distribution of nuclear property has yet been found.

3. The optimal loading pattern is a three-region bang-pang type which is very similar to that of the minimum fuel integral in the classical problem. The corresponding optimal control rod programming is unique and its policy is globally inner high and locally outer high.

4. Haling's principle does not give an optimal solution although it is near optimal and very useful for actual practice.

5. The multiregion control results in a burnup gain about three times larger than that of the two-region model.

Inclusion of the burnup dependence of the fission cross section, or the use of the FLARE type nodal equation with various reactivity feedbacks can further improve its practicality. Furthermore, the use of the space-energy-modal synthesis technique will make it possible to apply this method to fast breeder reactor core analyses.

APPENDIX

Coefficients a_{ji}, b_j, and c_i, and constraints for each variable of Eq. (13) are given below. In the following equations, $h = (\Delta x H/2M)^2$ and $N_1 = N - 1$. Note that the mesh point N_1 is the core edge.

$$
\left.
\begin{aligned}
&a_{m,1} = h\phi^0_{1,m} \qquad m = 1,\dots M\\
&a_{m,N(m-1)+L+1} = h(k^0_{01} - 1 - \alpha_1 e^0_{1,m} - u^0_{1,m}) - 1\\
&\qquad m = 1,\dots M\\
&a_{m,N(m-1)+L+2} = 1 \qquad m = 1,\dots M\\
&a_{m,N_1(m-1)+NM+L+1} = -h\alpha_1\phi^0_{1,m} \qquad m = 1,\dots M\\
&a_{M[Kn(i,l)-1]+m,Nu(m-1)+i+2NM+L-M} = -h\phi^0_{Kn(i,l),m}\\
&\qquad i = 1,\dots Nu\,,\quad l = 1,\dots Nd(i)\,,\quad m = 1,\dots M\\
&a_{M(n-1)+m,l} = h\phi^0_{n,m} \qquad m = 1,\dots M\,,\quad n = 2,\dots N_1\\
&\qquad l = 1\ (\text{if } L = 1)\,,\quad n\ (\text{if } L = N_1)\\
&a_{M(n-1)+m,N(m-1)+n+L-1} = 1 \qquad m = 1,\dots M\\
&\qquad n = 2,\dots N_1\\
&a_{M(n-1)+m,N(m-1)+n+L} = h(k^0_{0n} - 1 - \alpha_n e^0_{n,m} - u^0_{n,m}) - 2\\
&\qquad m = 1,\dots M\,,\quad n = 2,\dots N_1\\
&a_{M(n-1)+m,N(m-1)+n+L+1} = 1 \qquad m = 1,\dots M\\
&\qquad n = 2,\dots N_1\\
&a_{M(n-1)+m,N_1(m-1)+n+NM+L} = -h\alpha_n\phi^0_{n,m}\\
&\qquad m = 1,\dots M\,,\quad n = 2,\dots N\\
&a_{m+MN_1,N(m-1)+N+L-1} = \Delta x - 2l \qquad m = 1,\dots M\\
&a_{m+MN_1,N(m-1)+N+L} = 2l + \Delta x \qquad m = 1,\dots M\\
&a_{M(n-1)+MN+1,n+MN+L} = 1 \qquad n = 1,\dots N_1\\
&a_{M(n-1)+m+MN,N(m-2)+n+L} = -\Delta t\\
&\qquad m = 2,\dots M\,,\quad n = 1,\dots N_1\\
&a_{M(n-1)+m+MN,N_1(m-2)+n+NM+L} = -1\\
&\qquad m = 2,\dots M\,,\quad n = 1,\dots N_1\\
&a_{M(n-1)+m+MN,N_1(m-1)+n+NM+L} = 1\\
&\qquad m = 2,\dots M\,,\quad n = 1,\dots N_1\\
&a_{m+M(N+N_1),N(m-1)+n+L} = \Delta x\\
&\qquad m = 1,\dots M\,,\quad n = 1,\dots N_1
\end{aligned}
\right\}\ (\text{A.1})
$$

All the a_{ji}'s except those which appeared above are zeroes.

$$
\left.
\begin{aligned}
&b_m = [1 - h(k^0_{01} - 1 - \alpha_1 e^0_{1,m} - u^0_{1,m})]\phi^0_{1,m} - \phi^0_{2,m}\\
&\qquad m = 1,\dots M\\
&b_{M(n-1)+m} = -\phi^0_{n-1,m} + [2 - h(k^0_{0n} - 1 - \alpha_n e^0_{n,m} - u^0_{n,m})]\\
&\qquad \times\ \phi^0_{n,m} - \phi^0_{n+1,m}\\
&\qquad m = 1,\dots M\,,\quad n = 2,\dots N_1\\
&b_{m+MN_1} = (2l - \Delta x)\phi^0_{N_1,m} - (2l + \Delta x)\phi^0_{N,m}\\
&\qquad m = 1,\dots M\\
&b_{M(n-1)+MN+1} = -e^0_{n,1}\\
&b_{M(n-1)+m+MN} = -e^0_{n,m} + e^0_{n,m-1} + \Delta t\,\phi^0_{n,m-1}\\
&\qquad m = 2,\dots M\\
&b_{m+M(N+N_1)} = 1 - \Delta x \sum_{n=1}^{N_1} \phi^0_{n,m} \qquad m = 1,\dots M
\end{aligned}
\right\}\ (\text{A.2})
$$

$$
\left.
\begin{aligned}
&c_i = 1 \qquad i = 1,\dots L\\
&c_i = 0 \qquad i = L + 1,\dots 2NM + M(Nu - 1) + L
\end{aligned}
\right\}\ (\text{A.3})
$$

$$
\left.
\begin{aligned}
&\max(-\delta k, k_{\min} - k^0_{0n}) \le x_n \le \min(\delta k, k_{\max} - k^0_{0n})\\
&\qquad n = 1,\dots L\\
&\max(-\delta\phi, -\phi^0_{n,m}) \le x_{N(m-1)+n+L} \le \min(\delta\phi, f - \phi^0_{n,m})\\
&\qquad n = 1,\dots N_1\,,\quad m = 1,\dots M\\
&\max(-\delta\phi, -f - \phi^0_{N,m}) \le x_{N(m-1)+N+L} \le \min(\delta\phi, f - \phi^0_{N,m})\\
&\qquad m = 1,\dots M\\
&\max(-\delta e, -E) \le x_{N_1(m-1)+n+NM+L} \le \min(\delta e, E - e^0_{n,m})\\
&\qquad n = 1,\dots N_1\,,\quad m = 1,\dots M\\
&\max[-\delta u, -u^0_{Kn(i,1),m}] \le x_{Nu(m-1)+i+2NM-M+L}\\
&\qquad \le \min[\delta u, U - u^0_{Kn(i,1),m}]\\
&\qquad i = 1,\dots Nu\,,\quad m = 1,\dots M
\end{aligned}
\right\}\ (\text{A.4})
$$

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