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Burnup Optimization of Continuous Scattered Refueling

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A variational treatment of the burnup optimization of continuous scattered refueling is presented and numerical solutions are given for a slab reactor. It is made quantitatively clear how the reactor dimension, the xenon and the Doppler feedback reactivity, the burnup dependence of fission cross section and the reflector performance affect the power distribution that maximizes the average discharge exposure. Power flattening and burnup maximization are contradictory in general, but are consistent if, and only if, the condition of the perfect reflection at the core boundary is satisfied. The optimal power distribution is peaked in the central—and depleted in the outer region; and becomes flatter as the reflector performance is increased. The maximum average burnup depends on the burnup dependence of fission cross section and the strength of the Doppler and the xenon feedback reactivity, even if the average burnup calculated by the point-reactor model is the same. The former effect on the optimal power distribution is very small but the latter effects greatly contribute to power flattening. Both effects reduce the maximum burnup and the effects of the latter two are of comparable order. As the reactor becomes smaller, the maximum burnup decreases almost linearly to the neutron leakage. Optimal refueling has an advantage of more than 10% in the average burnup over the conventional flat-refueling rate method. However the difference from the flat-burnup method is very small, considering that the optimal refueling is handicapped by its very bad power distribution.

I. INTRODUCTION

With the arrival of economically competitive nuclear power, it has become very important to study the effect of the power distribution on the fuel burnup. One of the main tasks imposed on both designers and operators will be to make an effort to obtain the largest fuel burnup with the minimum fuel inventory within the various operational limitations.

The problem of finding the optimal control-rod programming to maximize the fuel burnup has been studied by Terney,[1] Suzuki,[2] and the author.[3] The result indicates that the burnup maximization and the power flattening are contradictory, and also the optimal power distribution is in general not constant throughout the reactor life. Suda[4] studied another problem of finding the optimal control-rod distribution to produce the largest power from the given core, showing that the power distribution is required to be flat in the central region of the core.

Besides these recent studies, there have been many works on the problem of the minimum critical mass,[5,6] and it has been shown that the neutron flux (more rigorously the fuel importance function) distribution should be kept flat to main-

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[2] A. SUZUKI and R. KIYOSE, *Trans. Am. Nucl. Soc.*, **11**, 441 (1968).
[3] H. MOTODA and T. KAWAI, *Nucl. Sci. Eng.*, **39**, 114 (1970).
[4] N. SUDA, T. YAMAGUCHI, and Y. SAKURAI, *J. Nucl. Sci. Technol.*, **5**, 452 (1968).
[5] G. GOERTZEL, *J. Nucl. Energy*, **2**, 193 (1956).
[6] M. OTSUKA, *Nucl. Sci. Eng.*, **18**, 514 (1964).

tain the criticality with the minimum fuel inventory in a thermal reactor. This may suggest that the maximum excess reactivity for fuel burnup, or the minimum refueling rate can be achieved by making the specific power distribution as flat as possible. However, this is not true, because to make a reactor critical with the minimum fuel mass is a static problem, whereas to obtain the maximum burnup is a dynamic one. The results obtained in the above-mentioned studies[1-3] suggest that it is required to burn the reactor fuels in such a way that the condition of the minimum critical mass can be attained as closely as possible at the end of the reactor life by the appropriate sequential withdrawal of control rods to get the maximum burnup.[a]

The aim of this paper is to consider this problem from the viewpoint of burnup maximization and to investigate the optimal distribution of power, flux, burnup, and refueling rate in a continuously refueled reactor, where no control rods are inserted and the criticality is maintained by appropriately controlling the refueling rate distribution.

Some characteristics of the continuous reactor refueling have already been studied extensively by Yasukawa.[7-9] He treated the following three types analytically using the elliptic functions: out-in, in-out, and bi-directional refueling. One critical velocity must be chosen to make a reactor just critical for a given fuel characteristic, and no optimization problem arises in the first two schemes. However, it was proved that the maximum burnup is obtained in the third scheme when the rate of fuel movement is equal in the two directions.

Another scheme of continuous refueling is here presented. This scheme can be called "scattered" or "graded" refueling, which has much larger freedom for optimization and is suitable for the present purpose.

II. STATEMENT OF THE PROBLEM

The continuous scattered refueling considered here is one which has the following property; the new fuels are charged at each point x in the core with the refueling rate $\omega(x)$, and the burnt fuels are discharged with the same rate and the corresponding burnup distribution $e(x)$ from the core, and no movement or replacement of partially burnt fuels are allowed.

The problem is to find the optimal distribution of power $p(x)$, neutron flux $\phi(x)$, refueling rate $\omega(x)$, and burnup $e(x)$ to maximize the average burnup, which can be defined as the discharge exposure averaged over the core, weighted with the refueling rate. This control problem can be put into a form suitable for the direct application of variational method.[10]

Before setting up the fundamental equations, the assumptions adopted in this treatment are given below.

1. The reactor considered is a one-dimensional slab reactor.

2. Effect of a reflector is treated by the boundary condition of the logarithmic derivative at the core edge.

3. One-group diffusion equation is assumed.

4. The burnup dependence of the infinite multiplication factor $k_\infty(e)$ and the macroscopic-fission cross section $\Sigma_f(e)$ is assumed to be linear.

5. Nuclear properties at each point are represented by the burnup-averaged quantities.

6. Only the xenon and the Doppler feedback are taken into account, and their power dependences are assumed to be of the form adopted in the FLARE code.[11]

7. No limitation is imposed on the maximum value of the power peaking factor but the consideration is taken to the metallurgical limit on the maximum discharge exposure.

In view of the foregoing assumption, the infinite multiplication factor k_∞ and the macroscopic-fission cross section Σ_f, and their averages $\overline{k}_\infty$ and $\overline{\Sigma}_f$ are expressed as Eq. (1):

$$\left.\begin{aligned} k_\infty(e) &= k_{\infty 0}(1 - ae) \quad , \quad \Sigma_f(e) = \Sigma_{f0}(1 - be) \\ \overline{k}_\infty &= \int_0^e k_\infty(e)de \Big/ \int_0^e de = k_{\infty 0}\left(1 - \frac{1}{2}ae\right) \\ \overline{\Sigma}_f &= \int_0^e \Sigma_f(e)de \Big/ \int_0^e de = \Sigma_{f0}\left(1 - \frac{1}{2}be\right) \end{aligned}\right\} \quad . \tag{1}$$

One-dimensional diffusion equation can therefore be written in the non-dimensional form as

[a]This is natural, since making the critical buckling minimum is equivalent to giving the maximum burnup in the usual situation.

[7]S. YASUKAWA, *Nucl. Sci. Eng.*, **24**, 239 (1966).
[8]S. YASUKAWA, *J. Nucl. Sci. Technol.*, **4**, 367 (1967).
[9]S. YASUKAWA, *Nucl. Sci. Eng.*, **35**, 1 (1969).

[10]L. D. BERKOVITZ, *J. Math. Anal. Appl.*, **3**, 145 (1961).
[11]D. L. DELP, D. L. FISHER, J. M. HARRIMAN, and M. J. STEDWELL, "FLARE—A Three-Dimensional Boiling-Water Reactor Simulator," GEAP-4598, General Electric Co. (1964).

$$4\left(\frac{M}{H}\right)^2 \frac{d^2}{dx^2}\phi(x) + \left[\overline{k}_\infty - 1 - \beta\overline{\Sigma}_f\phi(x) - \frac{\delta(1+\gamma)\overline{\Sigma}_f\phi(x)}{\overline{\Sigma}_f\phi(x)+\gamma}\right]\phi(x) = 0 \quad , \qquad (2)$$

where H is the ~~half~~-width of the core, β and δ are the Doppler and the xenon feedback reactivity at the rated power, and γ is a constant, such that $1 + \gamma$ is the ratio of saturated xenon reactivity at an infinitely large power to the average equilibrium value at the rated power. The reactor power is normalized so that the volume integral and the average rated power become 1.0,

$$\int_0^1 p(x)dx = 1.0 \quad , \qquad (3)$$

where $p(x) = \overline{\Sigma}_f(x)\phi(x)$. Migration area M^2 varies with burnups but its relative change is very small compared with that of $\overline{k}_\infty - 1$. Therefore, it can be considered constant in discussing Eq. (2), but the distinction is made between the coefficient a and b in Eq. (1). The boundary condition of Eq. (2) is given, assuming the symmetrical solution, as

$$\left.\begin{aligned} &\frac{d}{dx}\phi = 0 && \text{at } x = 0 \\ &\frac{d}{dx}\phi + \alpha\phi = 0 && \text{at } x = 1 \end{aligned}\right\} \quad , \qquad (4)$$

where α is the reciprocal extrapolated distance.

After an equilibrium state is reached, the following relation must hold among the refueling rate $\omega(x)$, the reactor power $p(x)$ and the burnup $e(x)$ (discharge exposure).

$$p(x) = \omega(x)e(x) \quad . \qquad (5)$$

Therefore, the average burnup $\overline{e}$ is calculated, weighted with the refueling rate ω as,

$$\overline{e} = \frac{\int_0^1 \omega(x)e(x)dx}{\int_0^1 \omega(x)dx} = \frac{1}{\int_0^1 [p(x)/e(x)]dx} \quad . \qquad (6)$$

The discharge exposure $e(x)$ must satisfy the following constraint imposed by the metallurgical limit.

$$e(x) \leq e_0 \quad . \qquad (7)$$

It is evident that the maximization of the average burnup is equivalent to the minimization of the total refueling rate, or the maximization of the fuel residence time when the total power output is fixed.

In discussing the effect of the reactor dimension, the Doppler and the xenon feedback reactivity universally the following quantities are used:

$$\left.\begin{aligned} A &= \frac{4(M/H)^2}{k_{\infty 0} - \beta - \delta - 1} \\ S_1 &= \frac{\beta}{k_{\infty 0} - \beta - \delta - 1} \\ S_2 &= \frac{\delta}{k_{\infty 0} - \beta - \delta - 1} \\ e_\infty &= \frac{2}{ak_{\infty 0}}(k_{\infty 0} - \beta - \delta - 1) \end{aligned}\right\} \quad . \qquad (8)$$

The numerator of A is proportional to the reactivity loss $M^2B_g^2$ by the leakage neutrons in a bare reactor and the denominator is the excess reactivity for fuel burnup. Thus, A represents the combined effect of a reactor dimension and a fuel characteristic. S_1 and S_2 represent the ratios of the Doppler and the xenon feedback reactivity to burnup reactivity, respectively. e_∞ is the discharge exposure which can be obtained in case of an infinitely large or a perfectly reflected reactor.

III. FORMULATION BY VARIATIONAL METHOD

The problem is to minimize the functional

$$J = e_\infty \int_0^1 \frac{p(x)}{e(x)}dx \qquad (9)$$

with the constraints of Eqs. (2), (3), and (7). In this problem it is arbitrary which variable to choose as the control variable $u(x)$ and here the reciprocal of burnup distribution, $e_\infty/e(x)$ is selected.

Thus, the constraints of Eqs. (3) and (7), and the functional J can be written as

$$\Sigma_{f0}\int_0^1 (1 - b'/u)\phi dx = 1 \qquad (10)$$

$$u \geq u_{\min} \equiv e_\infty/e_0 \qquad (11)$$

$$J = \Sigma_{f0}\int_0^1 (u - b')\phi dx \quad . \qquad (12)$$

The system equation is given from Eq. (2) as

$$\left.\begin{aligned} &\frac{d}{dx}\phi_1 = \phi_2 \\ &\frac{d}{dx}\phi_2 = \frac{1}{Ah}\left(\frac{C_1}{u_2} + \frac{C_2}{u} + C_3\right) \end{aligned}\right\} \quad , \qquad (13)$$

with the boundary conditions

$$\left.\begin{aligned} &\phi_2 = 0 && \text{at } x = 0 \\ &\phi_2 + \alpha\phi_1 = 0 && \text{at } x = 1 \end{aligned}\right\} \quad , \qquad (14)$$

where

$$\left.\begin{aligned}
\phi_1 &= \phi \\
C_1 &= -\Sigma_{f_0} b'\phi^2(1 - S_1\Sigma_{f_0} b'\phi) \\
C_2 &= \phi(\Sigma_{f_0}\phi + \Sigma_{f_0} b'\phi + \gamma) - S_1\Sigma_{f_0} b'\phi^2 \\
&\quad \times (2\Sigma_{f_0}\phi + \gamma - 1) - S_2\gamma\Sigma_{f_0} b'\phi^2 \\
C_3 &= S_1\phi(\Sigma_{f_0}\phi - 1)(\Sigma_{f_0}\phi + \gamma) + S_2\phi\gamma \\
&\quad \times (\Sigma_{f_0}\phi - 1) - (\Sigma_{f_0}\phi + \gamma)\gamma \\
h &= \Sigma_{f_0}\phi + \gamma - 1/u\, \Sigma_{f_0} b'\phi \\
\text{and} & \\
b' &= b/2\, e_\infty
\end{aligned}\right\} \quad . \tag{15}$$

Hamiltonian H can now be written as

$$\mathrm{H} = -\Sigma_{f_0}(u - b')\phi_1 + \lambda\left[\Sigma_{f_0}(1 - b'/u)\phi_1 - 1\right] + \psi_1\phi_2 + \frac{\psi_2}{Ah}\left(\frac{C_1}{u^2} + \frac{C_2}{u} + C_3\right) \quad , \tag{16}$$

where λ is the Lagrange multiplier which is introduced to cope with the constraint of Eq. (10), and ψ_1, ψ_2 are adjoint variables which are subjected to the following equations:

$$\left.\begin{aligned}
\frac{d}{dx}\psi_1 &= \Sigma_{f_0}(u - b') - \lambda\Sigma_{f_0}(1 - b'/u) - \frac{\psi_2}{Ah} \\
&\quad \times\left[\frac{1}{u^2}\frac{\partial C_1}{\partial\phi_1} + \frac{1}{u}\frac{\partial C_2}{\partial\phi_1} + \frac{\partial C_3}{\partial\phi_1} - \frac{\Sigma_{f_0}}{h}\right. \\
&\quad \left.\times\left(\frac{C_1}{u^2} + \frac{C_2}{u} + C_3\right)(1 - b'/u)\right] \\
\frac{d}{dx}\psi_2 &= -\psi_1
\end{aligned}\right\} \quad . \tag{17}$$

The boundary condition of Eq. (17) is given from Eq. (14) by the transversality condition as,

$$\left.\begin{aligned}
\psi_1 &= 0 \qquad \text{at } x = 0 \\
\psi_1 &= \alpha\psi_2 \quad \text{at } x = 1
\end{aligned}\right\} \quad . \tag{18}$$

Necessary condition for optimality requires that Hamiltonian be maximum with respect to u at each point of x. The u dependence of H is complicated but it can be shown that H is a concave function of u for the reasonable ranges of b', S_1, and S_2. Thus, the maximum condition becomes

$$\left.\begin{aligned}
&\Sigma_{f_0}\phi_1 u^2(u^2 - \lambda b') + \frac{\psi_2}{Ah}\left[(\Sigma_{f_0} b'\phi_1 + h)C_2 u^2\right. \\
&\quad \left.+ (\Sigma_{f_0} b'\phi_1 C_2 + 2C_1 h)u + \Sigma_{f_0} b'\phi_1 C_1\right] = 0 \qquad \text{if } \left.\frac{\partial \mathrm{H}}{\partial u}\right|_{u_{\min}} > 0 \\
&u = u_{\min} \qquad \text{if } \left.\frac{\partial \mathrm{H}}{\partial u}\right|_{u_{\min}} \leq 0
\end{aligned}\right\} \quad . \tag{19}$$

Six equations—(10), (13), (17), and (19) provide the solution for six unknown variables $\phi_1(x)$, $\phi_2(x)$, $\psi_1(x)$, $\psi_2(x)$, $u(x)$ and λ.

An accurate treatment of these equations is difficult because of the heavy nonlinearity, and these equations are not of suitable form to gain an insight into physical interpretation. Here some simplifications are made to see the effect of each parameter respectively, for which the numerical calculations are performed.

IV. SIMPLIFICATION

The simplest case is a bare slab reactor with no feedback. In this case $\alpha = \infty$, $S_1 = S_2 = 0$ and Eq. (19) reduces to

$$\left.\begin{aligned}
\lambda\Sigma_{f_0} b' - \frac{\psi_2}{A} &= u^2\Sigma_{f_0} \qquad \text{if } \left.\frac{\partial \mathrm{H}}{\partial u}\right|_{u_{\min}} > 0 \\
u &= u_{\min} \qquad \text{if } \left.\frac{\partial \mathrm{H}}{\partial u}\right|_{u_{\min}} \leq 0
\end{aligned}\right\} \quad . \tag{20}$$

First we will derive the equations without taking account of the limitation on u. By eliminating ψ_1 and ψ_2 from Eq. (17) and the first equation of Eq. (20), the following equations are obtained associated with the simplified form of Eq. (13):

$$\left.\begin{aligned}
A\frac{d^2}{dx^2}u^2 &= 2u - u^2 - [b' + \lambda(1 - b')] \\
A\frac{d^2}{dx^2}\phi &= (1/u - 1)\phi
\end{aligned}\right\} \quad . \tag{21}$$

The boundary condition of Eq. (21) is

$$\left.\begin{aligned}
\frac{d}{dx}\phi = 0 \quad &, \quad \frac{d}{dx}u = 0 \qquad \text{at } x = 0 \\
\phi = 0 \quad &, \quad u = (\lambda b')^{1/2} \quad \text{at } x = 1
\end{aligned}\right\} \quad . \tag{22}$$

Equations (10), (21), and (22) provide the optimal solution. It is evident that $(1 - 1/u)/A$ is the material buckling and $u \gtreqless 1$ means $k_\infty \gtreqless 1$, namely, $e(x) \lesseqgtr e_\infty$ corresponds to $k_\infty \gtreqless 1$. Equation for u is independent of ϕ and u is amenable to a solution of an elliptic function. The numerical solution shows that u is a monotonously decreasing function of x. Thus, the optimal distribution of u, considering the limitation on u, can be obtained by combining the solution of Eq. (21) and the second equation of Eq. (20) at the point x_0, where u attains its minimum allowable value $u_{\min}$, if $(\lambda b')^{1/2} < u_{\min}$. The resulting equations lead to

$$\left.\begin{aligned}
A\frac{d^2}{dx^2}u^2 &= 2u - u^2 - [b' + \lambda(1 - b')] \\
A\frac{d^2}{dx^2}\phi &= (1/u - 1)\phi \qquad \text{for } 0 \leq x \leq x_0 \\
A\frac{d^2}{dx^2}\psi &= (1/u_{\min} - 1)\psi + A\lambda(1 - b'/u_{\min}) \\
&\quad + A(b' - u_{\min}) \\
A\frac{d^2}{dx^2}\phi &= (1/u_{\min} - 1)\phi \qquad \text{for } x_0 \leq x \leq 1
\end{aligned}\right\} \quad . \tag{23}$$

The boundary condition of Eq. (23) is

$$\left.\begin{aligned} &\frac{d}{dx}\phi = 0 \ , \quad \frac{d}{dx}u = 0 && \text{at } x = 0 \\ &\psi = A(\lambda b' - u_{\min}^2) \ , && \\ &\frac{d}{dx}\psi = -2A\, u_{\min}\frac{d}{dx}u \ , \quad u = u_{\min} && \text{at } x = x_0 \\ &\phi = 0 \ , \quad \psi = 0 && \text{at } x = 1 \end{aligned}\right\} \ . \tag{24}$$

From these equations, the effect of reactor dimension and burnup dependence of Σ_f can be examined by systematically changing the value of A and b' for a reasonable range.

A little more improved simplification is to make $b' = 0$. This assumes the equivalence of neutron flux and power, which can be verified to be adequate by the solution of the above equations. This time the following coupled equations are obtained by the same procedure, in case that no limitation is taken into account on u.

$$\left.\begin{aligned} &A\frac{d^2}{dx^2}u^2 = 2u - \lambda - u^2\left\{1 + S_1(1 - 2\Sigma_{f0}\phi) + S_2\left[1 - (1+\gamma)\left(2 - \frac{\Sigma_{f0}\phi}{\Sigma_{f0}\phi + \gamma}\right)\frac{\Sigma_{f0}\phi}{\Sigma_{f0}\phi + \gamma}\right]\right\} \\ &A\frac{d^2}{dx^2}\phi = \left[1/u - 1 - S_1(1 - \Sigma_{f0}\phi) - S_2\left(1 - \frac{1+\gamma}{\Sigma_{f0}\phi + \gamma}\Sigma_{f0}\phi\right)\right]\phi \end{aligned}\right\} \ . \tag{25}$$

The boundary condition of Eq. (25) is

$$\left.\begin{aligned} &\frac{d}{dx}\phi = 0 \ , \quad \frac{d}{dx}u = 0 && \text{at } x = 0 \\ &\frac{d}{dx}\phi + \alpha\phi = 0 \ , \quad 2\frac{d}{dx}u + \alpha u = 0 && \text{at } x = 1 \end{aligned}\right\} \tag{26}$$

and Eq. (11) reduces to

$$\Sigma_{f0}\int_0^1 \phi\, dx = 1 \ . \tag{27}$$

Equations (25), (26), and (27) provide the optimal solution. This time Eq. (25) is coupled for ϕ and u, and analytical treatment is impossible. The following equations hold when the limitation on u is included:

$$\left.\begin{aligned} &A\frac{d^2}{dx^2}u^2 = 2u - \lambda - u^2\left\{1 + S_1(1 - 2\Sigma_{f0}\phi) + S_2\left[1 - (1+\gamma)\left(2 - \frac{\Sigma_{f0}\phi}{\Sigma_{f0}\phi + \gamma}\right)\frac{\Sigma_{f0}\phi}{\Sigma_{f0}\phi + \gamma}\right]\right\} \\ &A\frac{d^2}{dx^2}\phi = \left[1/u - 1 - S_1(1 - \Sigma_{f0}\phi) - S_2\left(1 - \frac{1+\gamma}{\Sigma_{f0}\phi + \gamma}\Sigma_{f0}\phi\right)\right]\phi \qquad \text{for } 0 \le x \le x_0 \\ &A\frac{d^2}{dx^2}\psi = \left\{1/u_{\min} - 1 - S_1(1 - 2\Sigma_{f0}\phi) - S_2\left[1 - (1+\gamma)\frac{\Sigma_{f0}\phi}{\Sigma_{f0}\phi + \gamma}\left(2 - \frac{\Sigma_{f0}\phi}{\Sigma_{f0}\phi + \gamma}\right)\right]\right\}\psi + A(\lambda - u_{\min}) \\ &A\frac{d^2}{dx^2}\phi = \left[1/u_{\min} - 1 - S_1(1 - \Sigma_{f0}\phi) - S_2\left(1 - \frac{1+\gamma}{\Sigma_{f0}\phi + \gamma}\Sigma_{f0}\phi\right)\right]\phi \qquad \text{for } x_0 \le x \le 1 \end{aligned}\right\} \ . \tag{28}$$

The boundary condition of Eq. (28) is

$$\left.\begin{aligned} &\frac{d}{dx}\phi = 0 \ , \quad \frac{d}{dx}u = 0 && \text{at } x = 0 \\ &\psi = -A\, u_{\min}^2 \ , && \\ &\frac{d}{dx}\psi = -2A\, u_{\min}\frac{d}{dx}u \ , \quad u = u_{\min} && \text{at } x = x_0 \\ &\frac{d}{dx}\phi + \alpha\phi = 0 \ , \quad \frac{d}{dx}\psi + \alpha\psi = 0 && \text{at } x = 1 \end{aligned}\right\} \ . \tag{29}$$

Effect of the Doppler and the xenon feedback and the reflector performance can be examined by changing the value of S_1, S_2, and α systematically for a reasonable range.

In both cases, Σ_{f0} can be fixed at 1.0 without loss of any generality.

V. APPROXIMATE ANALYSIS BY MODAL EXPANSION METHOD

It is another purpose of this paper to see how nonoptimal power distribution affects the average burnup. Therefore, an approximate analysis by modal expansion method is also employed. The constraint of Eq. (7) is not taken into account here. Neutron flux was expanded into a Fourier series up to 3 terms which is thought enough for the present purpose

$$\phi(x) = \sum_{n=1}^{3} a_n \cos \alpha_n x \quad . \tag{30}$$

Equation (4) requires that

$$\alpha_n \tan \alpha_n = \alpha \quad , \quad [(n-1)\pi \leq \alpha_n \leq (n-1/2)\pi] \ . \tag{31}$$

In the case when b' can be set at 0, a_1 can be expressed in terms of a_2 and a_3 by the normalization constraint of Eq. (27) with $\Sigma_{f0} = 1$.

$$a_1 = \frac{\alpha_1}{\sin \alpha_1}\left(1 - \frac{a_2}{\alpha_2}\sin \alpha_2 - \frac{a_3}{\alpha_3}\sin \alpha_3\right) \quad . \tag{32}$$

Thus, once a_2 and a_3 are specified, ϕ and $(d^2/dx^2)\phi$ are uniquely determined and the average burnup can be calculated by the direct integration as expressed in Eq. (33)

$$\frac{\bar{e}}{e_\infty} = \left(\int_0^1 \frac{\phi^2}{(1+S_1+S_2)\phi - S_1\phi^2 - S_2\dfrac{1+\gamma}{\gamma+\phi}\phi^2 + A\dfrac{d^2}{dx^2}\phi}\,dx\right)^{-1} \quad . \tag{33}$$

It can easily be shown that the Eq. (25) is the Euler-Lagrange equation of the functional Eq. (~~29~~ 33).

VI. METHOD OF NUMERICAL SOLUTION

Numerical solution by a digital computer is performed in each case. Equations (21), (23), (25), and (28) are two-point boundary value problems and some iterative method should be used to obtain the complete solution satisfying the boundary conditions at both sides and normalization constraint. Here, linearized iterative procedure is adopted.

With a given set of initial guesses $\phi(o)$, $u(o)$, and λ, transfer coefficients of small perturbations in these initial guesses to the changes of boundary condition at $x = 1$ and normalization constraint are calculated, and assuming that the effect of each perturbation is linearly independent, next improved guesses are calculated. This procedure is repeated until the convergence criteria are met. This method has provided good convergence, and most of the cases converged within 10 iterations. Integration step Δx and convergence criteria ϵ are set at 0.001 and 0.0005 for all cases, which assures the accuracy of 0.01% in the maximum average burnup $\bar{e}/e_\infty$ and 0.03% in the power peaking factor f. Simpson's integral formula is used in calculating Eq. (33) with the integration step Δx of 0.02, which assures the accuracy of 0.05% in $\bar{e}/e_\infty$.

VII. NUMERICAL RESULTS

A. *Effect of the Reactor Dimension and Burnup Dependence of* Σ_f

As can easily be understood by the definition of A and b', A is the reciprocal of non-dimensional material buckling of a fresh fuel and $2b'$ is the fraction of the decrement of fission cross section at $e = e_\infty$.

Typical values of A and b' of a large ATR (Advanced Thermal Reactor; boiling-light-water cooled, heavy-water-moderated reactor which employs on-power refueling) are approximately 0.02 and 0.2. Upper limit of A is $(2/\pi)^2$. Therefore, numerical calculations are performed for 0.02 ~ 0.2 of A and 0.0 ~ 0.3 of b'.

Some results are shown in Table I for the maximum average burnup $\bar{e}/e_\infty$ and the corresponding power peaking factor f, together with the maximum burnup $e_{max}\,[= \max_{0 \leq x \leq 1} e(x)]$, in case of the bare slab reactor with no reactivity feedback and no constraint on e. The maximum average burnup decreases with larger value of b' for each of A. This means that even if the average discharge exposure is the same when calculated by the area

TABLE I

Effect of Reactor Dimension and Burnup Dependence of Fission Cross Section on Maximum Average Burnup and Power Peaking Factor in Bare Slab Reactor

$S_1 = 0, \quad S_2 = 0, \quad e_0 = \infty$

(No limitation imposed on discharge exposure)

A	b'	e/e_∞	f	e_{max}/e_∞
0.02	0.0	0.9661	1.9725	∞
	0.1	0.9649	1.9346	3.1051
	0.2	0.9636	1.9245	2.1944
	0.3	0.9622	1.9132	1.7907
0.06	0.0	0.8715	1.7415	∞
	0.1	0.8696	1.7363	2.9487
	0.2	0.8676	1.7403	2.0826
	0.3	0.8658	1.7443	1.6987
0.10	0.0	0.7724	1.6667	∞
	0.1	0.7705	1.6750	2.7755
	0.2	0.7687	1.6839	1.9602
	0.3	0.7670	1.6927	1.5988
0.20	0.0	0.5209	1.6075	∞
	0.1	0.5197	1.6164	2.2802
	0.2	0.5187	1.6265	1.6099
	0.3	0.5178	1.6363	1.3134

method or the point reactor model, the actual maximum average burnup is smaller if the rate of fission cross-section decrease with fuel burnup is larger, although its effect is small and less than 1%.

Optimal distributions of power, neutron flux, burnup, and refueling rate are shown in Figs. 1 to 3 for $A = 0.02$. The optimal power distribution is peaked in the central—and depleted in the outer region. Physically this means that a higher burnup is attained by charging new fuels frequently in the central region where fuel importance is higher. Accordingly, the optimal burnup distribution increases monotonously toward the outer region, and vice versa for the optimal refueling-rate distribution. The degree of distortion in the power distribution is larger for smaller value of A. It

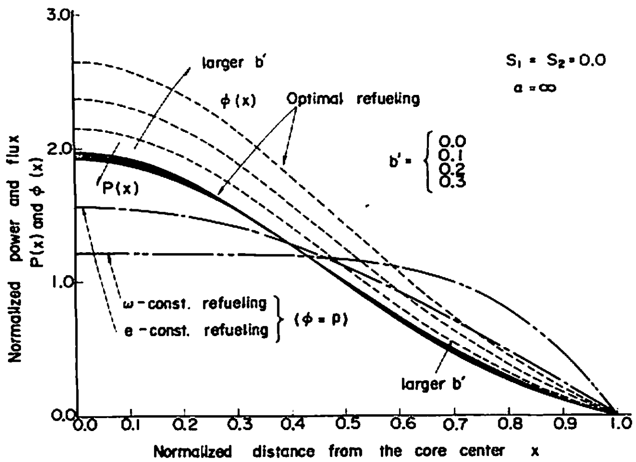

Fig. 1. Optimal power and flux distribution (Bare slab reactor with no feedback, $A = 0.02$, $e_0 = \infty$).

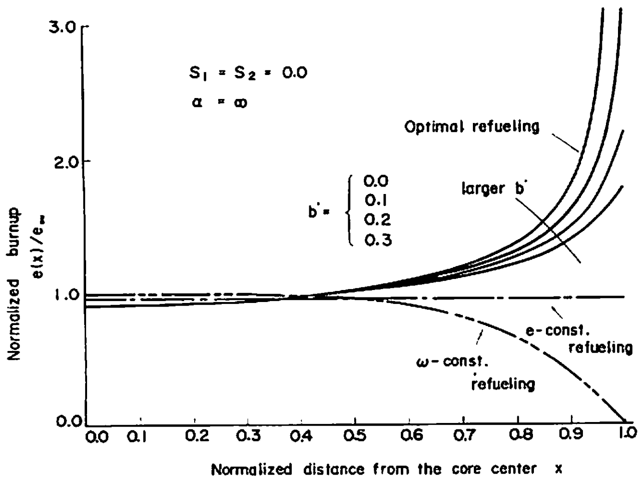

Fig. 2. Optimal burnup distribution (Bare slab reactor with no feedback, $A = 0.02$, $e_0 = \infty$).

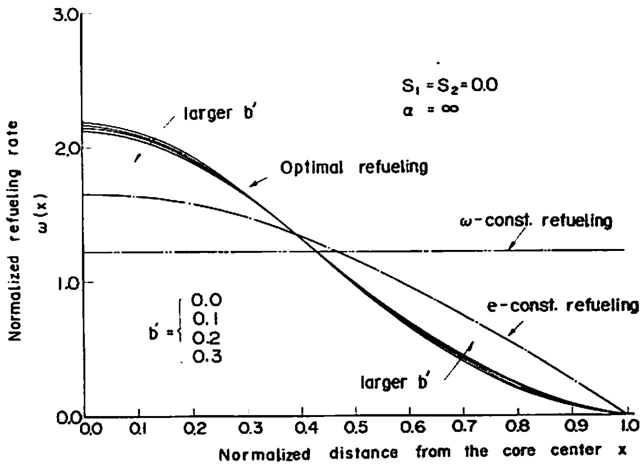

Fig. 3. Optimal refueling-rate distribution (Bare slab reactor with no feedback, $A = 0.02$, $e_0 = \infty$).

is evident that burnup optimization and power flattening are contradictory.

As A increases, freedom for control becomes smaller and in the limit where $A = (2/\pi)^2$, no freedom is left and only the cosine distribution is allowed for power and neutron flux.

The discharge exposure near the core boundary is very large and goes infinitely as b' becomes 0. This indicates the necessity of the inclusion of the limitation on the maximum allowable discharge exposure. Some results are presented in Table II, where the maximum allowable discharge exposure e_0/e_∞ is set at 1.5 and 1.3. No great changes can be shown compared with the values in Table I. The power peaking factor is improved a little, because the buckling in the outer region

TABLE II

Effect of Reactor Dimension and Burnup Dependence of Fission Cross Section on Maximum Average Burnup and Power Peaking Factor in Bare Slab Reactor

$S_1 = 0, \quad S_2 = 0, \quad e_0 = 1.5\,e_\infty, \quad 1.3\,e_\infty$

(Limitation imposed on discharge exposure)

		$e_0 = 1.5\,e_\infty$		$e_0 = 1.3\,e_\infty$	
A	b'	$\bar{e}/e_\infty$	f	$\bar{e}/e_\infty$	f
0.02	0.0	0.9658	1.9417	0.9652	1.9364
	0.1	0.9647	1.9336	0.9643	1.9280
	0.2	0.9635	1.9239	0.9633	1.9199
	0.3	0.9622	1.9131	0.9621	1.9110
0.10	0.0	0.7721	1.6643	0.7717	1.6628
	0.1	0.7702	1.6746	0.7701	1.6730
	0.2	0.7686	1.6838	0.7685	1.6830
	0.3	0.7670	1.6927	0.7669	1.6925

becomes constant. The reduction in the maximum average burnup is negligible, and the considerable changes in the burnup distribution near the core periphery do not give a large effect on the average burnup.

Optimal distribution of power, neutron flux, and burnup are shown in Fig. 4, corresponding to those in Figs. 1 and 2. No remarkable changes can be seen except that the burnup distribution $e(x)$ is kept its maximum allowable value e_0 near the outer edge of the core.

The effect of b' on the optimal power and refueling-rate distribution is small, on the whole, although considerable differences of burnup distribution are shown in the outer region when no constraint on $e(x)$ is taken into account. It can be said that the assumption of the equivalence of power and flux ($b' = 0$) is adequate because the optimal burnup distributions are very close to each other and flat in the central region where the weight of the refueling rate is large and furthermore the difference of burnup distributions in the outer region becomes much smaller for a reflected reactor, when the constraint on $e(x)$ is taken into account.

Qualitatively it can be said that the convexity of the power distribution in the outer region greatly reduces the neutron leakage and promotes the fuel burnup, but the detail of the optimal power distribution cannot be explained by this nature alone. If the neutron leakage is related one-to-one to fuel burnup, it is best to make $(d/dx)\phi$ at the core edge 0. Direct integration of Eq. (21) reads

$$[2u - u^2 - b' - \lambda(1 - b')]\phi - A\frac{d}{dx}u^2\frac{d}{dx}\phi = C_0 \ . \quad (34)$$

Assume that $(d/dx)\phi(1) = 0$, C_0 must be 0 because $\phi(1) = 0$, then $\phi = C_0'(d/dx)u^2$ must hold. This leads to $\phi(0) = 0$ because $(d/dx)\,u(0) = 0$ and is contradictory. Thus, it is proved that $(d/dx)\phi(1) \neq 0$ and there exists the neutron leakage even in the optimal solution. This fact will be referred to later in Part E.

Effect of the reactor dimension on the optimal fuel burnup $\bar{e}/e_\infty$ and the corresponding power peaking factor f is shown in Figs. 5 and 6. The optimal burnup decreases almost linearly to the neutron leakage for a given fuel characteristic or to the reciprocal of burnup reactivity of a fresh fuel for a given core size. This converges to 1.0 when A goes to 0. It should be noted that the variation of f is very sharp for small A contrary to the linear dependence of $\bar{e}/e_\infty$, and the effect of b' on f is reversed for small and large A. In the limit of $A = 0$, extrapolation suggests that f goes up to infinity, but in fact any value is allowed for f in this limit. This can be verified in Eq. (21). By setting $A = 0$, $u = 1$ and $\lambda(1 - b') + b' = 1$ are obtained and any distribution satisfying the boundary condition can be a solution to ϕ. The same arguments can be applied to the case where the constraint on e is taken into account.

B. Comparison with Other Refueling Methods

Two types of other simple refueling methods are calculated for comparison. These are refueling schemes where either distribution of burnup or refueling rate is flat and constant.

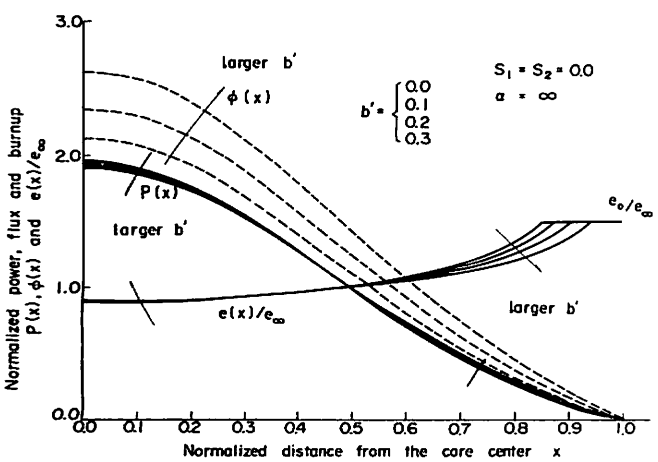

Fig. 4. Optimal power, flux and burnup distribution (Bare slab reactor with no feedback, $A = 0.02$, $e_0 = 1.5\,e_\infty$).

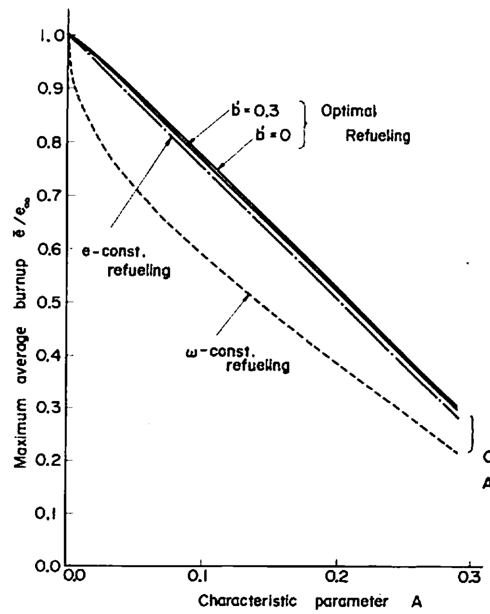

Fig. 5. Effect of reactor dimension on optimal fuel burnup (Bare slab reactor with no feedback, $e_0 = \infty$).

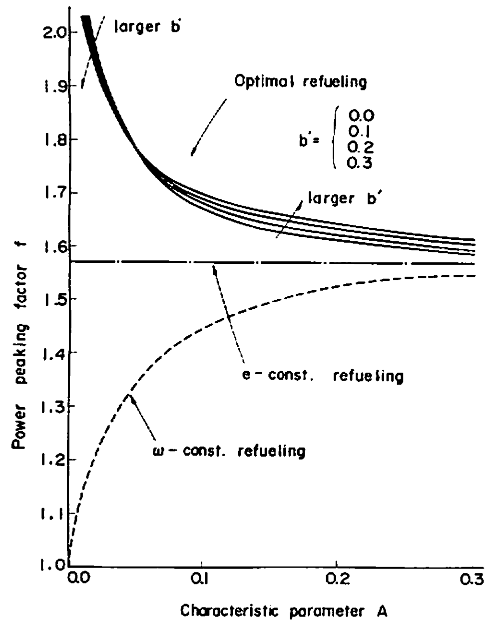

Fig. 6. Effect of reactor dimension on power peaking factor (Bare slab reactor with no feedback, $e_0 = \infty$).

In the former case, material buckling becomes constant and this results in the cosine distribution of flux and refueling rate. Criticality condition requires that

$$\frac{\overline{e}}{e_\infty} = 1 - \left(\frac{\pi}{2}\right)^2 A \quad . \tag{35}$$

In the latter case, ϕ is subjected to the following equation:

$$A \frac{d^2}{dx^2} \phi + (1 - B\phi)\phi = 0 \quad , \tag{36}$$

where B is the eigenvalue and equal to $\overline{e}/e_\infty$.

Characteristics of these refuelings are shown in Figs. 1 to 3, 5 and 6. Flux distribution is flattest in ω-constant refueling, but the average burnup is smallest of all. This situation can be understood from the burnup distribution in Fig. 2. The effect of burnup optimization has an advantage of 15% in the average discharge exposure over the conventional ω-constant refueling, but the difference from e-constant refueling is very small. This means that the simple e-constant refueling is near optimum and the merit of burnup optimization is not so great that in an actual core design we must always be careful of the balance among other factors such as the increase of fuel inventory due to the bad power distribution.

C. Effect of the Doppler and the Xenon Feedback

Succeeding analyses are performed for $b' = 0$, and no distinction is made between power and flux. Parameters A and γ are fixed at 0.02 and 0.3. The latter quantity takes the value ranging over 0.2 to 0.4 depending upon fuel enrichment. Extrapolated distance l is also fixed reasonably at 0.4 ($\alpha = 1/l = 2.5$). Standard values of S_1 and S_2 for ATR are 0.03 and 0.2. Therefore calculations are performed for $0.0 \sim 0.09$ of S_1 and $0.0 \sim 0.6$ of S_2.

Results are shown in Table III for $\overline{e}/e_\infty$, f, and e_{max}. The maximum average burnup decreases with larger values of S_1 and S_2. It should be kept in mind that to vary the values of S_1 and S_2 without changing the value of A, means that the average burnup is always conserved in a point-reactor sense by changing either fuel enrichment or core size. Therefore, the effect of S_1 and S_2 on the maximum average burnup is entirely due to the effect of power distribution control. Stronger feedback inevitably promotes better power flattening and thus decreases the maximum average burnup. The effect of the Doppler feedback on the optimal distribution of power, burnup, and refueling rate is shown in Fig. 7, and the effect of the xenon feedback is shown in Fig. 8. The power distribution is here again larger in the central region than in the outer region, although both feedbacks considerably contribute to power flattening.

TABLE III

Effect of Doppler and Xenon Feedback on Maximum Average Burnup and Power Peaking Factor in Reflected Slab Reactor

$A = 0.02, \quad \alpha = 2.5, \quad l = 0.4, \quad e_0 = \infty$

(No limitation imposed on discharge exposure)

S_1	S_2	$\overline{e}/e_\infty$	f	e_{max}
0.00	0.0	0.9810	1.6223	1.3278
	0.2	0.9765	1.4581	1.2567
	0.4	0.9735	1.3509	1.2151
	0.6	0.9714	1.2791	1.1878
0.03	0.0	0.9752	1.4172	1.2321
	0.2	0.9727	1.3259	1.1971
	0.4	0.9709	1.2636	1.1741
	0.6	0.9695	1.2192	1.1574
0.06	0.0	0.9719	1.3026	1.1800
	0.2	0.9703	1.2487	1.1607
	0.4	0.9690	1.2094	1.1466
	0.6	0.9680	1.1798	1.1355
0.09	0.0	0.9697	1.2341	1.1475
	0.2	0.9686	1.1996	1.1358
	0.4	0.9676	1.1731	1.1264
	0.6	0.9669	1.1521	1.1188

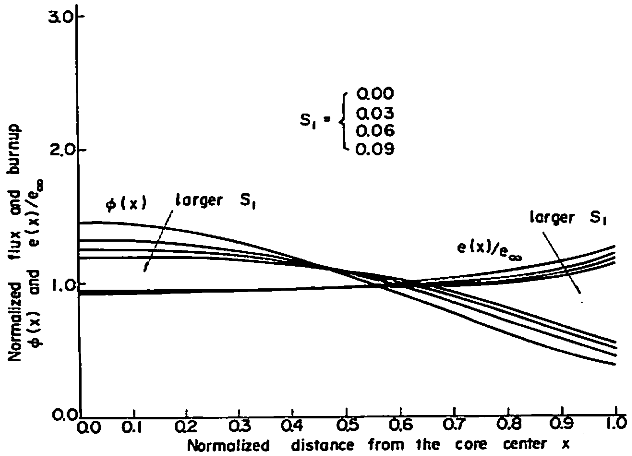

Fig. 7. Effect of Doppler feedback on optimal flux and burnup distribution (Reflected slab reactor, $A = 0.02$, $\alpha = 2.5$, $S_2 = 0.2$, $b' = 0$).

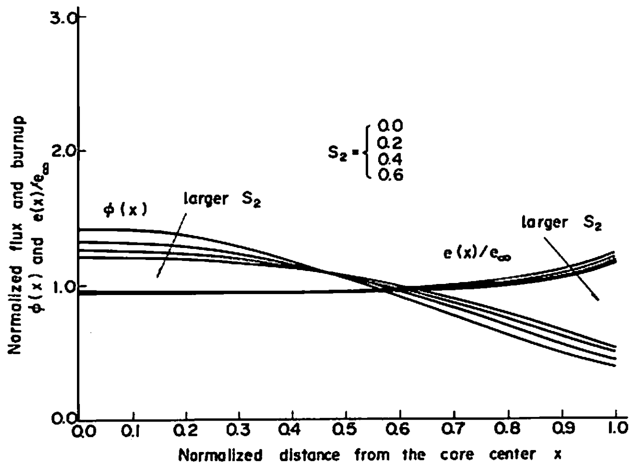

Fig. 8. Effect of xenon feedback on optimal flux and burnup distribution (Reflected slab reactor, $A = 0.02$, $\alpha = 2.5$, $S_1 = 0.03$, $b' = 0$).

Existence of these feedbacks reduces the maximum burnup by about 1% and power peaking by about 20% in the present ATR. The effect of both feedbacks on the maximum average burnup and the power peaking factor is of comparable order, respectively, although S_2 is larger by an order of magnitude than S_1. This is because the xenon feedback saturates with large power, contrary to the fact that the Doppler feedback is approximately proportional to power. In Fig. 9, both effects on $\bar{e}/e_\infty$ and f are shown in contour lines. The relation between S_1 and S_2, which gives the same burnup individually, is almost linear and both effects are additive. The constraint on e is not necessary for $l = 0.4$.

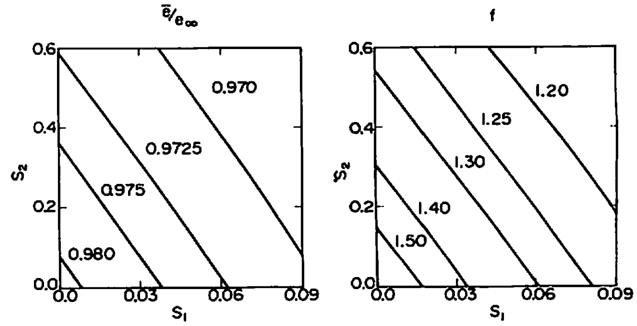

Fig. 9. Effect of Doppler and xenon feedback on maximum average burnup and corresponding power peaking factor (Reflected slab reactor, $A = 0.02$, $\alpha = 2.5$, $b' = 0$).

D. *Effect of the Reflector Performance*

In order to see the effect of the reflector performance, extrapolated distance l is varied from 0 to ∞. S_1 and S_2 are fixed here at 0.03 and 0.2.

Numerical results are shown in Table IV for $\bar{e}/e_\infty$, f, and e_{max}. The maximum average burnup monotonously increases as the reflector performance is increased and vice versa for the corresponding power peaking factor, both approaching 1.0.

The optimal distributions of power, burnup, and refueling rate are shown in Figs. 10 and 11 for various values of l (or α). Each of the three distributions becomes flatter as l goes to infinity and flat distribution is obtained regardless of the magnitude of feedbacks when the condition of the perfect reflection is satisfied. Burnup maximization and power flattening is consistent if, and only if, this condition is satisfied. The effect of reflec-

TABLE IV

Effect of Reflector Performance on Maximum Average Burnup and Power Peaking Factor

$A = 0.02$, $S_1 = 0.03$, $S_2 = 0.2$, $e_0 = \infty$

(No limitation imposed on discharge exposure)

α	l	$\bar{e}/e_\infty$	f	e_{max}
∞	0.00	0.9427	1.6993	∞
100.0	0.01	0.9441	1.6652	4.4137
10.0	0.10	0.9550	1.5420	1.6841
5.0	0.20	0.9629	1.4468	1.3795
2.5	0.40	0.9727	1.3259	1.1971
1.6667	0.60	0.9785	1.2541	1.1311
1.0	1.00	0.9849	1.1748	1.0774
0.5	2.00	0.9914	1.0971	1.0376
0.2	5.00	0.9963	1.0412	1.0147
0.0	∞	1.0000	1.0000	1.0000

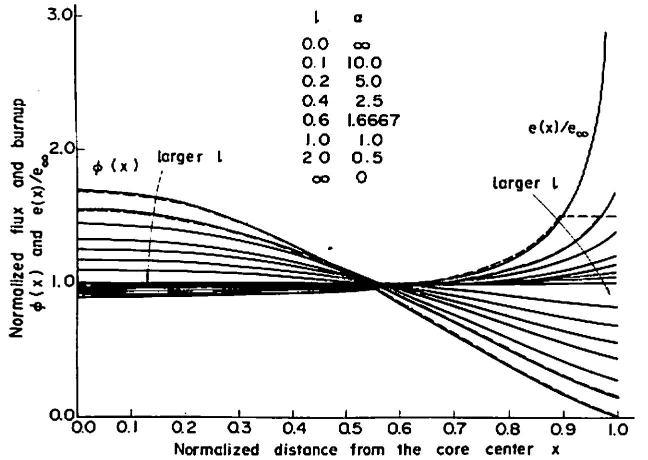

Fig. 10. Effect of reflector performance on optimal flux and burnup distribution (Reflected slab reactor, A = 0.02, S_1 = 0.03, S_2 = 0.2, b' = 0, $e_0 = \infty$ and 1.5 e_∞).

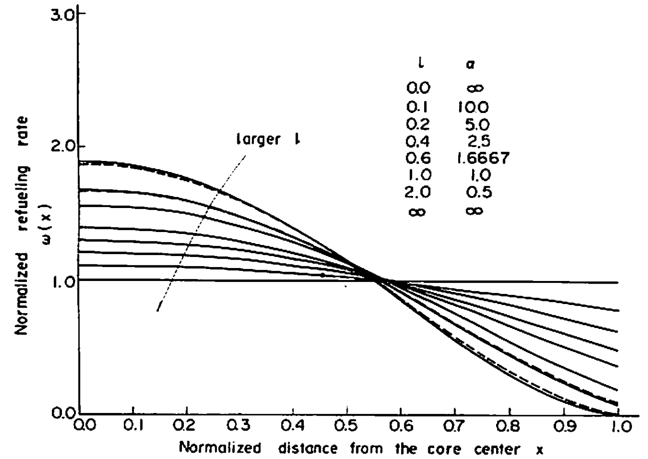

Fig. 11. Effect of reflector performance on optimal refueling rate distribution (Reflected slab reactor, A = 0.02, S_1 = 0.03, S_2 = 0.2, b' = 0, $e_0 = \infty$ and 1.5 e_∞).

tor tends to saturate with l larger than 1.0. Optimal reflector performance should be decided by taking into account those conflicting effects: the reduction of fuel cycle cost by the better fuel burnup and the increase of capital cost by the improved reflector performance.

Usual analysis in the problem of minimum critical mass has been to find the condition for reactivity to be stationary, and it is equivalent to maximize the total recovery of reactivity by suitably rearranging the fuel distribution. On the other hand, the burnup optimization requires another condition, namely, the minimization of the decrease in reactivity by fuel burnup as well as the maximization of the recovery of reactivity by refueling, and these are equivalent to the maximization of the average fuel residence time. Therefore, the burnup decreases, if the specific power is forced to be flat in the central region.

Table V shows the effect of the maximum allowable discharge exposure e_0/e_∞ on the maximum average burnup $\bar{e}/e_\infty$ and the power peaking factor f. Here again, the decreases in $\bar{e}/e_\infty$ and f are very small and negligible.

TABLE V

Effect of Maximum Allowable Discharge Exposure on Maximum Average Burnup and Power Peaking Factor

A = 0.02, S_1 = 0.03, S_2 = 0.2

α	l	e_0/e_∞	$\bar{e}/e_\infty$	f
∞	0.0	2.0	0.9424	1.6823
∞	0.0	1.5	0.9422	1.6813
∞	0.0	1.4	0.9420	1.6802
∞	0.0	1.3	0.9418	1.6792
∞	0.0	1.2	0.9413	1.6761
10.0	0.1	1.5	0.9550	1.5418
10.0	0.1	1.4	0.9549	1.5411
10.0	0.1	1.3	0.9548	1.5395
10.0	0.1	1.2	0.9545	1.5353

E. *Effect of the Power Distribution on the Average Burnup*

In this section the relation between the power distribution and the average burnup is discussed.

Contour lines of power peaking factor for the bare slab reactor are shown in Fig. 12 associated with the power distributions at the typical values of a_2 and a_3, the coefficients of the 2nd and the

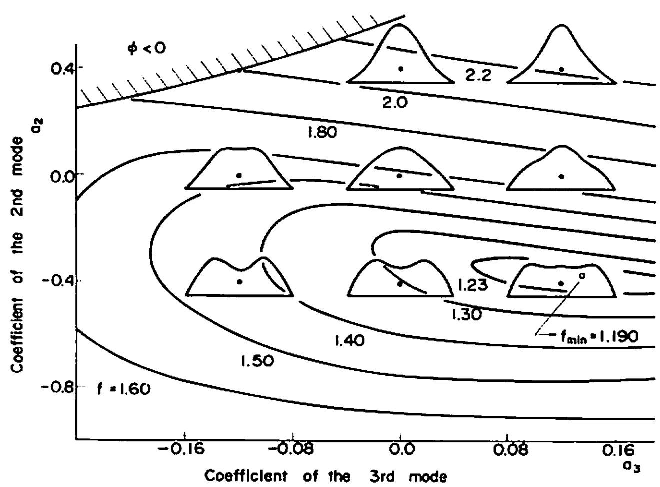

Fig. 12. Contour lines of power peaking factor and flux distribution (Bare slab reactor, $\alpha = \infty$, b' = 0).

3rd mode. It can be seen that considerably wide range of power distribution can be calculated by only the three mode expansion. Contour lines of the average burnup are shown in Figs. 13 and 14 for two values of A: 0.02 and 0.1 in case of no feedback. It can be understood that the power distribution that maximizes the average burnup is very bad for each of A, but still worse distribution reduces the burnup conversely. As A becomes larger, the region of negative burnups $[\phi + A(d^2/dx^2)\phi < 0]$ becomes wider and in the limit of $A = (2/\pi)^2$, only $a_2 = a_3 = 0$ is allowed where the power distribution is cosine and the average burnup is 0.

The dotted curve in Fig. 13 is a trajectory connecting the tangent points of each contour lines of the average burnup and the power peaking factor. In another word, this is a group of states where the average burnup is maximum for the given power peaking factor or the power peaking factor is minimum for the given average burnup. The optimal design point should be decided somewhere on this curve reflecting upon the balance between the two conflicting factors of the fuel burnup and the fuel inventory.

The situation is completely the same for a reflected reactor with xenon and Doppler reactivity feedbacks.

Comparison of modal expansion method with variational method is made in Figs. 15 and 16 for two cases. Their agreement is fairly good except for the burnup distribution near the core edge.

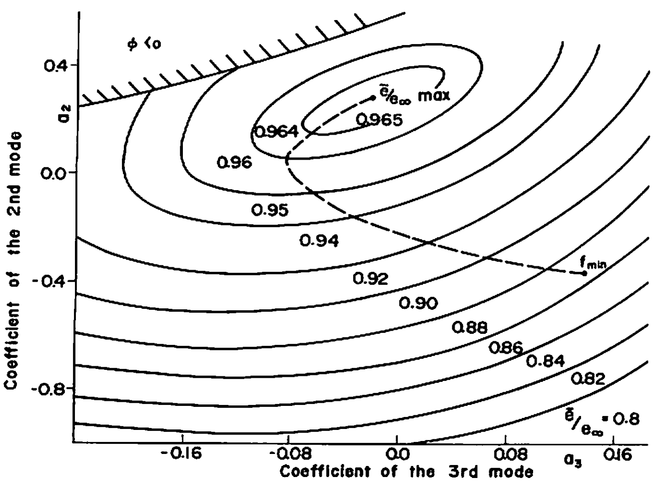

Fig. 13. Contour lines of average burnup (Bare slab reactor, $A = 0.02$, $\alpha = \infty$, $S_1 = S_2 = 0.0$, $b' = 0$).

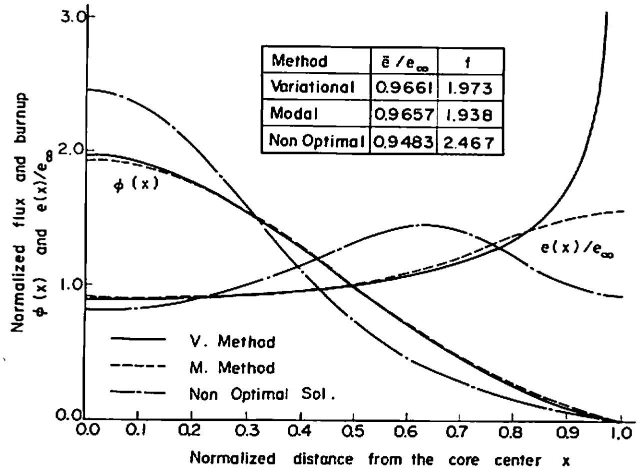

Fig. 15. Comparison of modal expansion method with variational method ($A = 0.02$, $b' = 0$, $S_1 = S_2 = 0.0$, $\alpha = \infty$, $e_0 = \infty$).

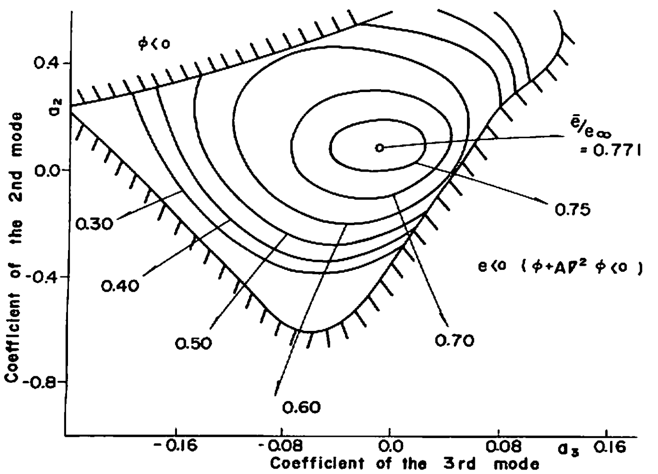

Fig. 14. Contour lines of average burnup (Bare slab reactor, $A = 0.10$, $\alpha = \infty$, $S_1 = S_2 = 0.0$, $b' = 0$).

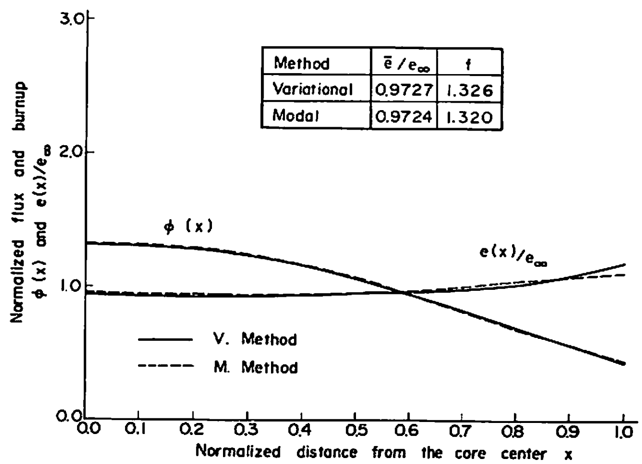

Fig. 16. Comparison of modal expansion method with variational method ($A = 0.02$, $b' = 0$, $S_1 = 0.03$, $S_2 = 0.2$, $\alpha = 2.5$, $e_0 = \infty$).

This large difference is understandable considering that the expansion is terminated at the third term. An example of a nonoptimal solution by modal expansion method is also shown in Fig. 15 for comparison. The neutron leakage from the core is smaller in this case than the optimal solution, but the average burnup is about 2% less. This supports the argument discussed in Part A.

VIII. CONCLUSION

The problem of the burnup optimization of continuous scattered refueling is presented. A relation between power distribution and the average discharge exposure is formulated. The equations that the optimal flux and burnup distributions are subjected to are derived by variational method and numerical solutions are given for one-dimensional slab reactor, from which the following conclusions are obtained:

1. The burnup optimization and the power flattening are contradictory in a usual situation and consistent if, and only if, the reflector performance is perfect.

2. Optimal power distribution is peaked in the central—and depleted in the outer region of the core. This nature does not change even when the effects of the Doppler and the xenon reactivity feedback and the reflector are taken into account.

3. Optimal burnup distribution increases monotonously toward the core edge and, if the constraint on e is to be violated, it is kept at its upper limit e_0 thereafter.

4. As the core size and/or the excess reactivity for fuel burnup decreases, the maximum average burnup decreases almost linearly to the neutron leakage calculated by the geometrical buckling.

5. Even if the average burnup calculated by the point-reactor model is the same, the maximum average burnup depends on (a) the burnup dependence of fission cross section, and (b) the strength of the Doppler and the xenon feedback. It becomes smaller for larger values of b', S_1, and S_2.

6. The effect of the reflector performance on the maximum average burnup is remarkable, but tends to saturate with the extrapolated length larger than 1.0.

7. Optimal refueling has an advantage of more than 10% in the average burnup over the conventional flat-refueling rate method. However, the difference from the flat-burnup method is very small, considering that the serious sacrifice is paid for the bad power distribution.

This information is useful in planning the refueling scheme in a reactor employing on-power refueling. However, in applying these results to an actual core design, more detailed study on local power peaking due to the finiteness of the number of fuel assemblies is necessary.

It is also felt necessary to include a constraint imposed on the maximum value of power peaking factor f for a reactor with poor reflector performance, which is left for further extension of this study.

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