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A Theory of Control-Rod Programming Optimization in Two-Region Reactors

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INTRODUCTION

Recent development of a computer and modern control theory has made it possible to handle, successfully, the various problems of optimal controls in nuclear reactors. Control-rod programming and fuel-management optimization are very important problems, since power generating cost is mainly determined by the hot spot factor and the fuel burnup. Wall[1] treated the latter problem and Terney[2] the former, both using the method of dynamic programming.[3]

The aim of this note is to try to interpret the coupled effect of the control-rod programming and the fuel burnup, geometrically, by the trajectory drawn in the burnup space. Some conventional rod programmings are discussed and compared. The problem of optimizing the control-rod programming to maximize the fuel burnup with the constraint imposed on the hot spot factor is formulated and solved for a typical boiling water reactor (BWR). The method of Maximum Principle[4,5] is applied to the two-region bare reactor; and it has been assumed that the radial and the axial power distribution are separable within the validity of one-group diffusion approximation.

THEORY

It is convenient to introduce a mathematical concept, "burnup space," in discussing a relation among control-rod programming, fuel burnup, and fuel management. A state of nuclear reactor with K regions is uniquely determined in one-group approximation by one representative nuclear property σ_k averaged over each region, assuming one quantity is necessary and sufficient enough for this

[1]I. WALL and H. FENECH, *Nucl. Sci. Eng.*, **22**, 285 (1965).
[2]W. B. TERNEY and H. FENECH, *Trans. Am. Nucl. Soc.*, **11**, 354 (1968).
[3]R. BELLMAN, *Dynamic Programming*, Princeton Univ. Press, Princeton, New Jersey (1957).
[4]L. S. PONTRYAGIN, V. G. BOLTYANSKII, R. V. GAMKRELIDZE, and E. F. MISHCHENKO, *The Mathematical Theory of Optimal Processes*, Interscience Publishers, New York (1962).
[5]L. D. BERKOVITZ, *J. Math. Anal. Appl.*, **3**, 145 (1961).

purpose. Define Σ space or burnup space as a K dimensional space with σ_k as its k'th coordinate. Then the change of a reactor state $\boldsymbol{\Sigma}(t)$ with fuel burnup can be expressed as a trajectory in this space.

Neglecting the variation of diffusion coefficient with fuel burnup, the diffusion equation can be written in the simple dimensionless form

$$\nabla^2 \phi(\boldsymbol{r}) + S_k \phi(\boldsymbol{r}) = 0 \ , \qquad 0 \leq \| \boldsymbol{r} \| \leq 1 \ , \tag{1}$$

where $S_k = \sigma_k - C_k$. Here, σ and C are non-dimensional nuclear properties which can be considered as "material buckling" of fuels and control rods, respectively, but are related to Eq. (1) only in the form of the difference $S = \sigma - C$.

Criticality condition, $G(\boldsymbol{S}) = 0$, gives a $K-1$ dimensional hyper-surface (critical surface) in the K dimensional Σ space. This surface divides the space into super- and sub-critical regions. Control-rod distribution "C" can be expressed as a vector in Σ space. It must be chosen so that a vector $\boldsymbol{S} = \boldsymbol{\Sigma} - \boldsymbol{C}$ is on the critical surface. If a point $\boldsymbol{S}$ is determined somewhere on the critical curve, the flux $\boldsymbol{\Phi}$, averaged over each region, can be uniquely determined; and the flux time increment $\delta\Theta$ for the small time interval δt, is just $\boldsymbol{\Phi}\delta t$, which causes the corresponding nuclear property change $\delta\boldsymbol{\Sigma}$.

The dynamics of this system is given as

$$\left.\begin{aligned} \boldsymbol{\Sigma} &= \boldsymbol{H}(\Theta) \\ \frac{d\Theta}{dt} &= \boldsymbol{\Phi}(\boldsymbol{S}) \\ \boldsymbol{S} &= \boldsymbol{\Sigma} - \boldsymbol{C} \\ G(\boldsymbol{S}) &= 0 \ . \end{aligned}\right\} \tag{2}$$

The farther the point $\boldsymbol{\Sigma}$ is located above the critical surface, the more degrees of freedom for operation can be obtained because the possible operationable region becomes wider.

APPLICATION

Some Discussions on the Conventional Rod Programmings

For simplicity, a symmetrical bare-slab reactor with two regions of equal volume is considered with additional assumptions,

a. equivalence of neutron flux and power

b. linear depletion of nuclear property with fuel burnup.

In this case Eq. (1) reduces to the simplest form

$$\frac{d^2\phi}{dx^2} + S_k \phi = 0 \qquad \begin{aligned} k &= 1 \text{ for } 0 \leq x \leq 0.5 \\ k &= 2 \text{ for } 0.5 \leq x \leq 1.0 \ , \end{aligned} \tag{3}$$

where

$$\begin{aligned} S_k &= \sigma_k - C_k \\ \sigma_k &= [H^2/(4D)]\,(\nu\Sigma_f \epsilon p - \Sigma_a)_k \\ C_k &= [H^2/(4D)]\,\Sigma_{ck} \ . \end{aligned}$$

Criticality condition is given as Eq. (4),

$$\sqrt{-S_1}\tanh(0.5\sqrt{-S_1}) = -\sqrt{-S_2}\coth(0.5\sqrt{-S_2}) \ , \tag{4}$$

from which the average flux ratio $g \equiv \bar{\phi}_2/\bar{\phi}_1$ and the hot spot factor $f \equiv \max\phi/\text{av}\,\phi$ can be determined uniquely as

$$\left.\begin{aligned} g &= -\frac{S_1[\cosh(0.5\sqrt{-S_2}) - 1]}{S_2\cosh(0.5\sqrt{-S_2})} \\ f &= \begin{cases} \dfrac{\sqrt{S_1}}{(1+g)\sin(0.5\sqrt{S_1})} & S_1 > 0 \\ \dfrac{\sqrt{-S_1}\coth(0.5\sqrt{-S_1})}{(1+g)\sin(0.5\sqrt{S_2})} & S_1 \leq 0 \ . \end{cases} \end{aligned}\right\} \tag{5}$$

An example of burnup space is shown in Fig. 1.

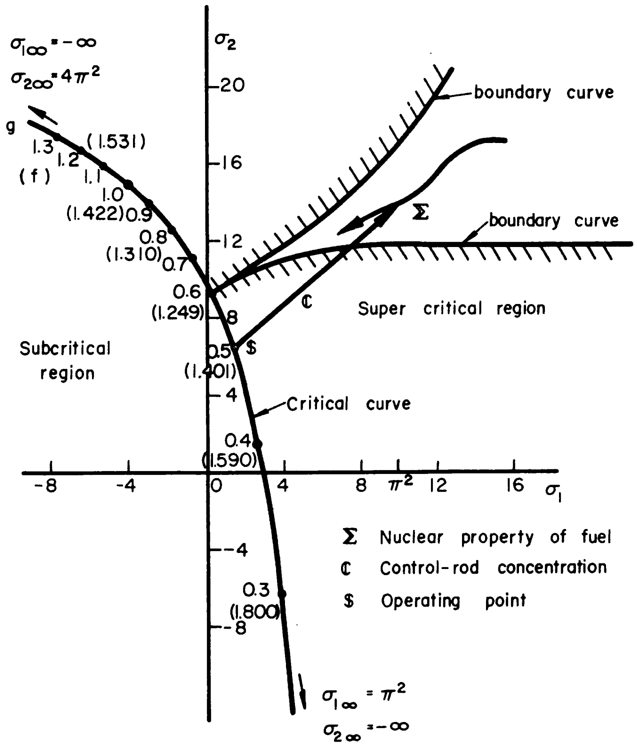

Fig. 1. An example of burnup space.

1. Uniform Control Method

This is a case $C_1 = C_2$, which is often assumed in burnup calculations. Operating point $\boldsymbol{S}$ is determined by the intersection of a 45° line drawn from the state $\boldsymbol{\Sigma}$ to the critical curve. The gradient of the trajectory is determined by g corresponding to $\boldsymbol{S}$. Repeating this procedure, it finally reaches the critical curve where $\boldsymbol{C} = 0$, which means the reactor has just burnt out. Various trajectories are depicted in Fig. 2.

The farther the starting point is located from the critical curve, the greater is the change of the power shape during operation. Because non-dimensional $\boldsymbol{\Sigma}$ is proportional to the square of core size H, this control method is not favorable for a large reactor.

2. Constant-Power Ratio Method

This is the case in which operating point $\boldsymbol{S}$ does not change throughout the reactor life, and the trajectory is a straight line, usually ending above the critical curve with one of the control rods withdrawn. Average burnup is determined by the decrement of the nuclear property $\delta\sigma$ averaged over the whole core, giving a larger burnup for the smaller value of $\sigma_1 + \sigma_2$.

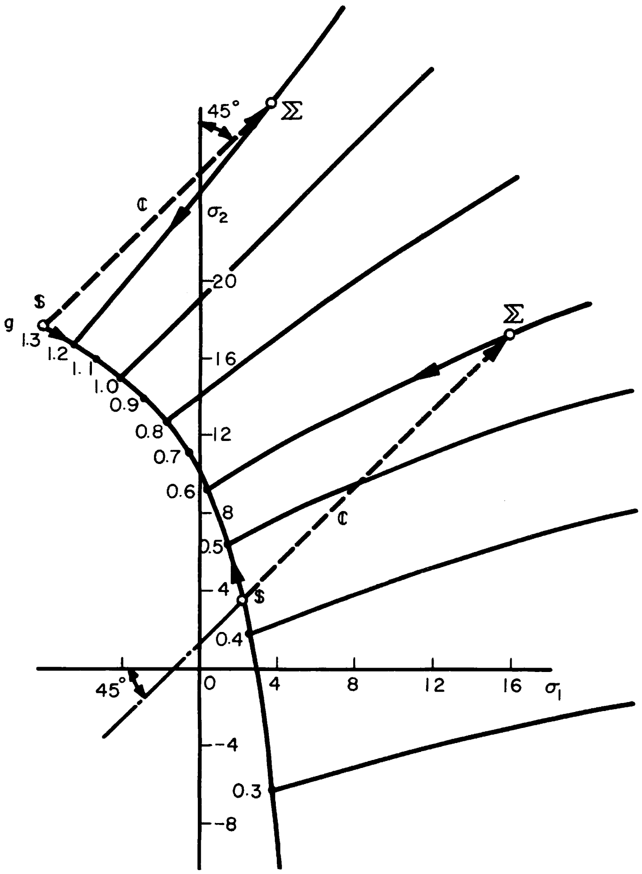

Fig. 2. Trajectories by uniform control method.

Three groups of trajectories starting from three different initial states are illustrated in Fig. 3. The maximum attainable region is uniquely specified for each of the initial states, shown as the dotted curves, and each state on these curves is meaningful in the sense that it gives the maximum burnup for a given power ratio (hot spot factor). One final state that gives the maximum burnup is the one that ends on the critical curve.

3. Consistent-Power Method

It has been shown that a specific value of the power ratio should be chosen to obtain the maximum burnup with the above constant-power ratio method. This is a so-called consistent power distribution. This trajectory can easily be determined by choosing an end point on the critical curve, first, and drawing a line in the opposite direction with the gradient corresponding to it, which is also shown in Fig. 3.

Whether the power shape is preferable or not, is entirely determined by the initial state itself, and since power ratios are not the same for reactors of different sizes, even with the same nuclear property, some difficulty may again be encountered in applying this method to a very large reactor.

4. Point Burnup

In zero-dimensional treatment, distribution of the power and the nuclear properties within a core are ignored and only the leakage is taken into account in the form of DB^2.

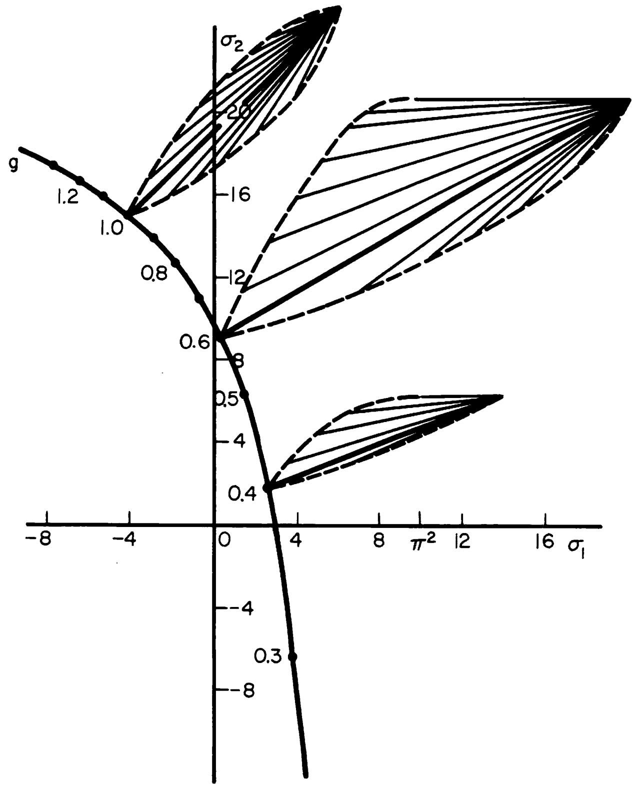

Fig. 3. Trajectories by constant-power ratio method.

This is the case $\sigma_1 = \sigma_2$ throughout the reactor life and the corresponding trajectory PQ_1 is illustrated in Fig. 4. The assumption of the uniform power distribution means $g = 1$. This is a special case of the constant-power ratio method, and the trajectory must end at Q_2, where $C_2 = 0$.

Other trajectories are also shown in Fig. 4 for comparison. No limitation is imposed on the maximum allowable hot spot factor. The trajectory PQ_5 that gives maximum burnup is the boundary curve, itself, which is explained in the next section. There are not so many differences in burnup but it is clear that point burnup usually overestimates fuel exposure.

It can be said for all the methods mentioned above that the control-rod programming has great influence on power shape in a large reactor, although there are not so many differences in the maximum burnup. Therefore, stress should rather be laid on the control of the power shape.

Optimal Rod Programming

It is possible to define the maximum allowable bounds of directions of the trajectory at any point in Σ space using the full freedom of control, which is shown in Figs. 1 and 4 as the boundary curves. Any points within this region are controllable. Therefore, it is possible to formulate the optimization problem of burnup maximization with the constraint imposed on the maximum allowable value of the hot spot factor f_{max}.

A typical BWR comparable to Vermont Yankee is considered. Assuming the separability of the radial- and axial-power distribution, the simple one dimensional treatment, above mentioned, becomes possible. Average fluxes are normalized so that the volume integral becomes

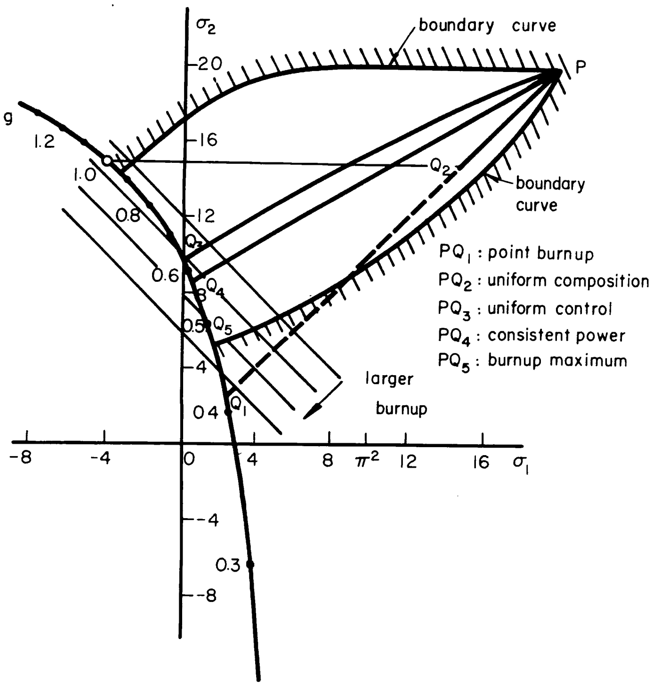

Fig. 4. Comparison of conventional rod programmings.

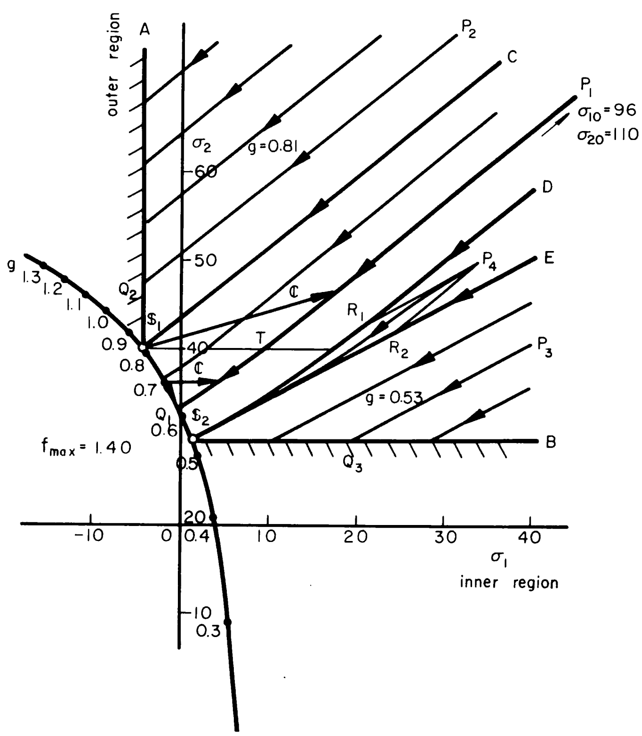

Fig. 5. Optimal trajectories for burnup maximization (radial two-region cylindrical reactor).

1.0. Two types of reactivity feedback are considered in the axial control rod optimization. One is the feedback from void and another is the feedback proportional to the power itself, and the method of Maximum Principle is used as the optimization technique.

1. Control-Rod Optimization for the Radial Power Distribution

With the assumption previously made, the problem is reduced to the maximization of the reactor life,

$$J \underset{\min}{=} - \int_0^{t_f} dt \quad . \tag{6}$$

Cylindrical geometry is used, and both $\sigma_2 - C_2$ and $\bar{\phi}_1$ are fitted to the second-order polynominals of $\sigma_1 - C_1$ for the sake of easy computation. The requirement, $f(t) \leq f_{\max}$, $C_1(t) \geq 0$ and $C_2(t) \geq 0$, gives the constraint of the form, $U_1(\sigma_1, \sigma_2) \leq C_1(t) \leq U_2(\sigma_1, \sigma_2)$ which can generally be written as $R(\sigma_1, \sigma_2, C_1) \geq 0$.

The dynamics of this system and the necessary condition for optimality are given as Eqs. (7) and (8),

$$\left.\begin{aligned} &H = 1 + \alpha(\Psi_1 - \Psi_2) \sum_{n=0}^{2} b_n(\sigma_1 - C_1)^n + 2\alpha\Psi_2 \\ &\frac{d\sigma_i}{dt} = \frac{\partial H}{\partial \Psi_i} \ , \quad \frac{d\Psi_i}{dt} = - \left[\frac{\partial H}{\partial \sigma_i} + \mu(t)\frac{\partial R}{\partial \sigma_i}\right] \\ &\sigma_2 - C_2 = \sum_{n=0}^{2} a_n(\sigma_1 - C_1)^n \end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned} &\frac{\partial H}{\partial C_1} + \mu(t)\frac{\partial R}{\partial C_1} = 0 \ , \quad \mu(t)R = 0 \ , \quad \mu \geq 0 \\ &H[\boldsymbol{\Psi}, \boldsymbol{\sigma}^*, C_1^*(t)] = \sup H[\boldsymbol{\Psi}, \boldsymbol{\sigma}^*, C_1(t)] \\ &H[\boldsymbol{\Psi}(t_f), \boldsymbol{\sigma}(t_f), C_1(t_f)] = 0 \\ &\boldsymbol{\Psi}(t_f) \cdot d\boldsymbol{\sigma}(t_f) = 0 \ . \end{aligned}\right\} \tag{8}$$

Optimal trajectories with $f_{\max} = 1.40$ and $\alpha = -34.4$ are given in Fig. 5. In order to burn out with both of the rods withdrawn, the initial state must be within the region CS_1S_2D, and the corresponding trajectory P_1TQ_1 shows that the inner region should be depleted as much as possible, giving the worst hot spot factor $f_{\max}$ at S_1, until the outer rod becomes fully withdrawn, and then the power shape gradually shifts into the inner region, while the reactor is maintained just critical with the inner rod only. The degree of depletion and the power shape of EOL (end of life) depends on the value, $f_{\max}$, and the initial state Σ. Corresponding optimal rod programming is shown in Fig. 6. Trajectories within the region DS_2E are not unique, and those within the region AS_1C and ES_2B show the constant power operation, both giving the worst hot spot factor $f_{\max}$ at S_1 and S_2, respectively.

2. Control-Rod Optimization for the Axial Power Distribution

Taking the two types of feedback mentioned above into account, S_k can be written as

$$S_k = \sigma_k - \beta\bar{\phi}_k - \gamma\bar{U}_k - C_k \ . \tag{9}$$

Doppler and xenon feedbacks are assumed to be proportional to the power. Thus β is the combined reactivity coefficient of both effects, and γ is the void coefficient. Both of them are appropriately normalized. Void distribution $u(x)$, therefore, its regionwise averages $\bar{U}_k$, can be calculated by using the pre-established relation between u and quality X which can be obtained by integrating the flux to the point x as

$$\begin{aligned} X &= X_{\text{in}} + (X_{\text{out}} - X_{\text{in}}) \int_0^x \phi \, dx \\ u(x) &= F(X) \ . \end{aligned} \tag{10}$$

By considering a fictitious criticality relation $G'(S_1', S_2') = 0$, where $S_k' = S_k + \beta\bar{\phi}_k + \gamma\bar{U}_k$, the problem is

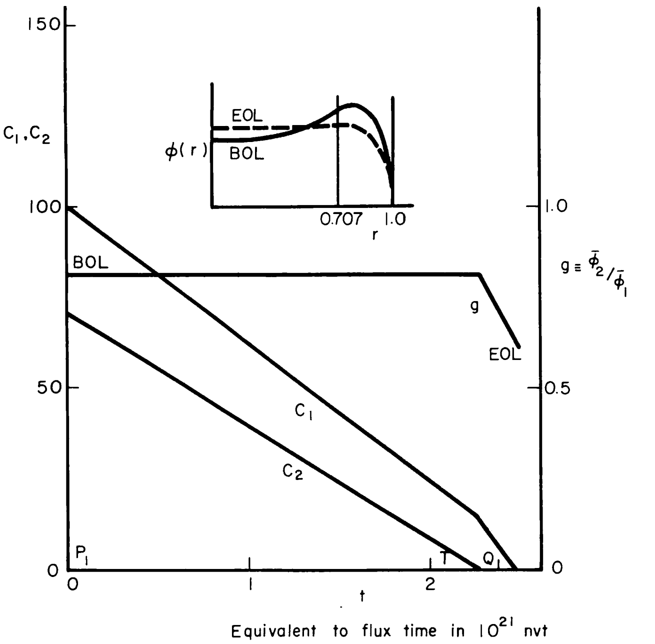

Fig. 6. Optimal radial control-rod programming and flux ratio (corresponds to the trajectory P_1TQ_1 in Fig. 5).

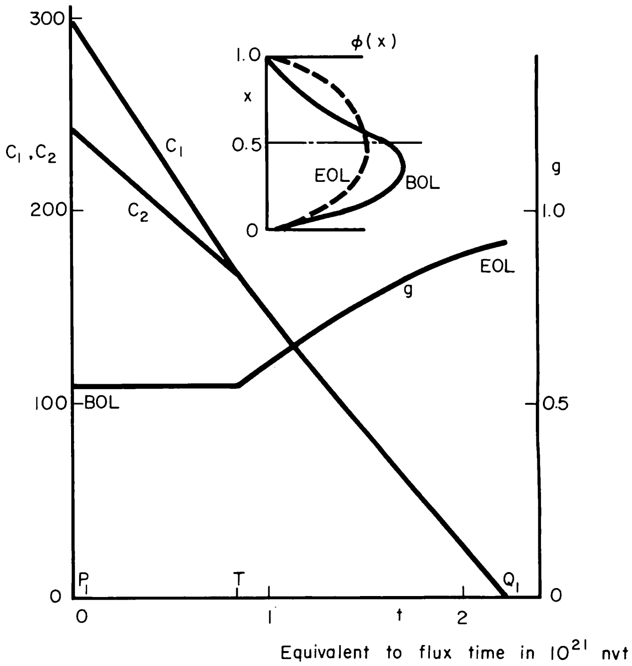

Fig. 8. Optimal axial control-rod programming and flux ratio (corresponds to the trajectory P_1TQ_1 in Fig. 7).

reduced to the one with no feedback. Another constraint added is $C_1(t) \geq C_2(t)$ which comes from the fact that control rods are inserted from the bottom in BWR.

Optimal trajectories with $f_{max} = 1.80$, $\alpha = -120$, $\beta = 50$, and $\gamma = 120$ are given in Fig. 7. Optimal control within the region $CQ_3S_2'D$ requires the constant-power ratio operation at S_2', giving the worst hot spot factor f_{max}, followed by the uniform control operation. Corresponding optimal rod programming is shown in Fig. 8. Trajectories within the region $AS_1'Q_3C$ and $DS_2'B$ show the uniform control and the constant-power operation respectively, the latter giving the worst hot spot factor f_{max}. Every trajectory indicates that the power shape of the upper part should be depleted as badly as possible within the constraint $f \leq f_{max}$ and $C_1 \geq C_2 \geq 0$.

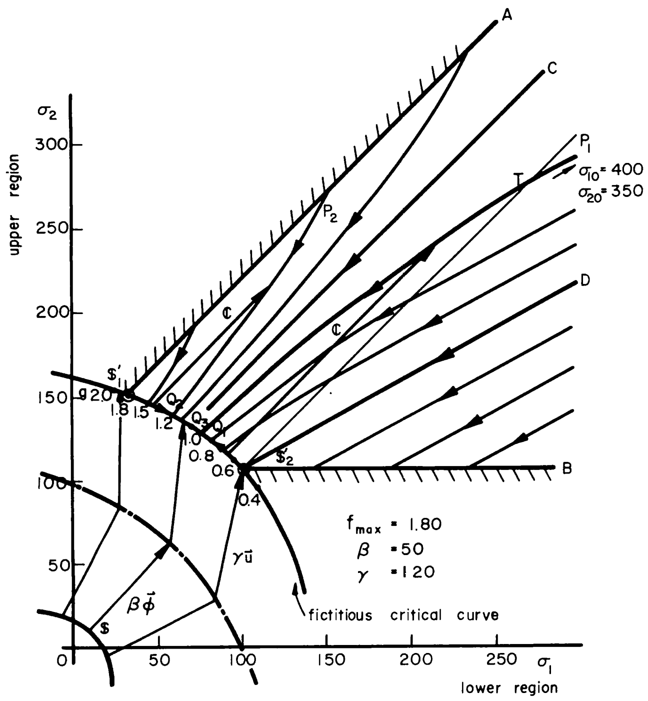

Fig. 7. Optimal trajectory for burnup maximization (axial two-region slab reactor).

CONCLUSION

By introducing a concept of phase-space analysis into burnup problems; criticality relation, power and control-rod distribution, and fuel burnup have been given geometrical meaning./ Some ~~methods~~ conventional rod programming are discussed, and the optimal rod programming to maximize the fuel burnup is obtained for a typical BWR. The result indicates that the burnup optimization and the power flattening are evidently contradictory, and that the power shape should be depleted as badly as possible in the inner and upper region of the core within the constraints imposed on the hot spot factor and the control-rod distribution.

We should like to note, lastly, that this method can be directly extended to find the optimal refueling scheme associated with its optimal rod programming in an equilibrium cycle. In this case, some relations hold between the initial and final state, and it is better to formulate a problem of finding the optimal rod programming that minimizes the maximum hot spot factor during the refueling interval. Some results have already been obtained with the more realistic model by using the technique of dynamic programming.

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